Single-Peakedness and Total Unimodularity for Multiwinner Elections

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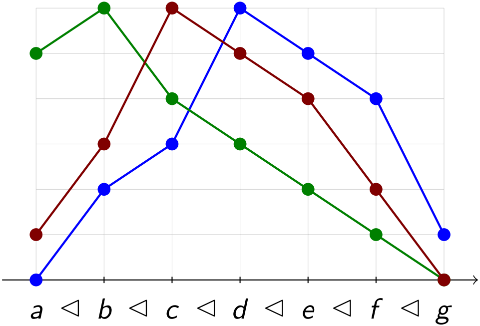
Multiwinner Elections

- **Input:** Finite set N of agents and set C of candidates; for each agent $i \in N$ a preference relation $\succcurlyeq_i$ over C.
- **Output:** A *committee* $W \subseteq C$ of exactly k candidates.
- So need a voting rule $f : \mathcal{R}(C)^N \to \{W \subseteq C : |W| = k\}$.
- **Applications:** Proportional parliaments, representative committees, shortlisting, movies on airplanes, ...
- **Problem:** Many popular rules optimise an objective function over all $\binom{m}{k}$ committees $W \rightsquigarrow$ NP-hard winner determination.

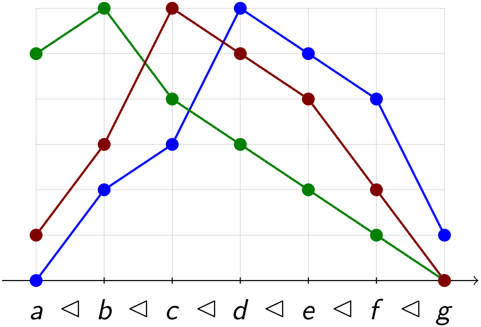
Structured Preferences

- One way to deal with hardness: look for tractable classes of profiles.
- Various forms of *structure* are promising
- Main example: *single-peaked* preferences
- Alternative space is one-dimensional, e.g. numerical quantities



Single-Peaked Preferences

- Suppose alternative space admits a left-to-right ordering $\lhd$.
- Then $\succcurlyeq_i$ is single-peaked w.r.t. $\lhd$ if whenever $\text{top}(i) \lhd a \lhd b$ or $b \lhd a \lhd \text{top}(i)$, then $a \succcurlyeq_i b$.
- A profile is *single-peaked* if there is some $\lhd$ such that each $\succcurlyeq_i$ is single-peaked w.r.t. $\lhd$.
- **Fact:** Can decide efficiently whether a profile is single-peaked.



Chamberlin–Courant rule

- One popular multiwinner voting rule is due to political scientists Chamberlin and Courant (1983)
- **Idea:** Every voter is *represented* by their favourite candidate in W, and obtains corresponding utility
- $\rightsquigarrow$ return committee with highest utilitarian welfare
- Committee $W \subseteq C$ gets objective value

$$\sum_{i \in N} \max\{\text{Borda-score}_i(c) : c \in W\}.$$

- NP-complete to find optimum committee.

Chamberlin–Courant rule: Single-Peaked

- Betzler, Slinko, Uhlmann (JAIR 2013): CC is easy to compute when preferences are single-peaked.
- Dynamic Programming algorithm:
  1. Find axis $\lhd$ witnessing single-peakedness.
  2. Build up committee from left to right.
  3. In DP table, compute how much we gain by adding c to committee, by tracking who would be represented by c.

Proportional Approval Voting

- **Criticism of CC:** Only gain utility from *one* committee member, ignore rest.
- Alternative rule: **PAV**, working with approval ballots (dichotomous preferences)
- **Idea**: voters gain utility from each committee member that they approve, but *decreasing marginal returns*.
- PAV assigns committee $W \subseteq C$ the objective value

$$\sum_{i \in N} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|W \cap v_i|}.$$

- NP-complete to find optimum committee.
- Can use non-harmonic weights, but these give good *proportionality* properties (Aziz et al., SCW 2016).

PAV: Structure

- For approval ballots, single-peaked means "approve an interval"
- Elkind and Lackner (IJCAI 2015) study whether PAV becomes easy for single-peaked preferences
- **Problem:** The Betzler et al. algorithm does not work for PAV – voters can have many "representatives"
- The Betzler et al. algorithm works if voters only approve few candidates – then there are few representatives (Elkind and Lackner 2015)
- Elkind and Lackner also find tractablility for some further restricted subclasses (e.g., when every voter approves one of the extreme candidates)
- Elkind and Lackner **conjecture**: PAV remains NP-hard for general single-peaked preferences.

PAV: Integer Programming Formulation

$$
\begin{aligned}
\text{maximise} \quad & \sum_{i \in N} \sum_{\ell \in [k]} \frac{1}{\ell} \cdot x_{i,\ell} && \text{(PAV-IP)} \\
\text{subject to} \quad & \sum_{c \in C} y_c \ = \ k && (2) \\
& \sum_{\ell \in [k]} x_{i,\ell} \leqslant \sum_{i \text{ approves } c} y_c \qquad \text{for } i \in N && (3) \\
& x_{i,\ell} \ \in \ \{0,1\} \qquad \text{for } i \in N,\ \ell \in [k] && (4) \\
& y_c \ \in \ \{0,1\} \qquad \text{for } c \in C
\end{aligned}
$$

Total Unimodularity

- A $\{-1,0,1\}$-matrix A is called *totally unimodular* (TU) if for every square submatrix B of A has $\det B \in \{-1,0,1\}$.
- If A is TU, then the integer program

$$\max c^T x \text{ subject to } Ax \leqslant b, x \in \mathbb{Z}^n \qquad \text{(IP)}$$

  is solved optimally by its LP relaxation $\rightsquigarrow$ poly-time solvable.
- Every matrix with the *consecutive ones property* is TU.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

PAV, Single-Peaked, TU

- **Theorem:** If preferences are single-peaked, then (PAV-ILP) is TU and hence poly-time solvable.
- *Example:* 4 candidates a, b, c, d, committee size $k = 2$
  Voter 1 approves $\{a, b, c\}$, and Voter 2 approves $\{c, d\}$.
  Single-peaked: $a \lhd b \lhd c \lhd d$.

$$
\begin{aligned}
\text{maximise} \quad & (x_{1,1} + \tfrac{1}{2}x_{1,2}) + (x_{2,1} + \tfrac{1}{2}x_{2,2}) && \text{(PAV-IP')} \\
\text{subject to} \quad & y_a + y_b + y_c + y_d = 2 && (2) \\
& x_{1,1} + x_{1,2} \leqslant y_a + y_b + y_c && (3) \\
& x_{2,1} + x_{2,2} \leqslant y_c + y_d && (3) \\
& \text{all variables binary}
\end{aligned}
$$

$$
A_{\text{PAV-IP'}} = \begin{array}{c} \begin{matrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} & y_a & y_b & y_c & y_d \end{matrix} \\ \left[\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \end{matrix}\right] \end{array}
$$

Other rules

- For **CC**, one can also design an ILP formulation that becomes TU if preferences are single-peaked.

$$
\begin{aligned}
\text{maximise} \quad & \sum_{i \in N} \sum_{r \in [m]} x_{i,r} \\
\text{subject to} \quad & \textstyle\sum_{c \in C} y_c = k \\
& x_{i,r} \leqslant \textstyle\sum_{c \,:\, \mathrm{rank}(c) \geqslant r} y_c \qquad i \in N,\ r \in [m] \\
& x_{i,r}, y_c \in \{0,1\}
\end{aligned}
$$

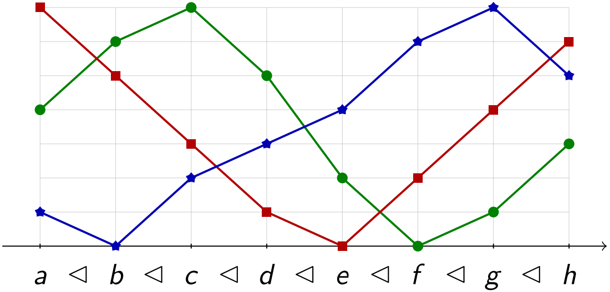
- This gives a poly-time algorithm for the single-peaked case, but it is not a *specialised* algorithm: it works also if the input is not single-peaked.
- This algorithm does not first need to find an axis $\lhd$.
- Note that the input preferences are encoded in the *constraints*, not the objective function (as would be more natural).

Other rules

- The same approach works for generalisations of CC via ordered weighted averages: *OWA-based rules* (Skowron et al. AIJ 2016).
- For example, this includes the "t-Borda rules", in which voters are represented by their t favourite committee members.
- Also works for egalitarian variants of CC and PAV.

Single-Peaked on Circles

- Suppose alternatives are arranged in a *circle*.
- Preferences are single-peaked on this circle (SPOC) if for every voter, we can cut the circle so that the vote is single-peaked on the resulting line (Peters and Lackner, AAAI 2017).
- The TU approach can also be made to work for SPOC profiles: CC, PAV, etc. remain tractable for this larger domain.



Conclusions and Future Work

- New method of designing algorithms for single-peaked settings
- Are there other problems for which this works?
- Open: PAV with "voter interval" preferences (Elkind and Lackner 2015)
- Open: Characterise class of profiles for which the ILPs shown are TU.
- Approach allows to optimise *any* linear objective
  $\rightsquigarrow$ application to certain facility location problems on a line.

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