

Distribution Rules Under Dichotomous Preferences

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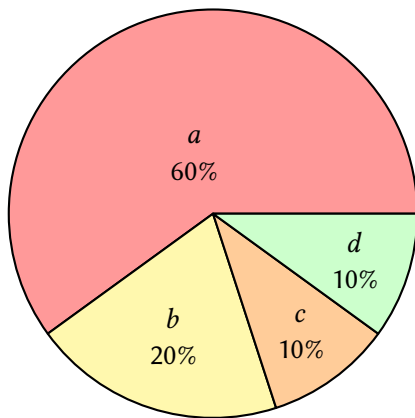
2023-10-10

Conference on Voting Theory and Preference Aggregation
Celebrating Klaus Nehring's 65th Birthday

Based on a paper presented at the ACM EC Conference 2021

Distribution rules

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Model

- ▶ Set of voters, $N = \{1, \dots, n\}$.
- ▶ Set of projects $A = \{x_1, \dots, x_m\}$.
- ▶ Possible outcomes $\Delta(A) = \{p : A \rightarrow [0, 1] : \sum_{x \in A} p_x = 1\}$.
- ▶ Each voter $i \in N$ approves projects $A_i \subseteq A$.
- ▶ Voter gets utility $u_i(p) = \sum_{x \in A_i} p_x$ from distribution p .
- ▶ Voting rule takes the approval sets and outputs a distribution.

A. Bogomolnaia, H. Moulin, and R. Stong. “Collective choice under dichotomous preferences”. In: *Journal of Economic Theory* 122.2 (2005), pp. 165–184

C. Duddy. “Fair sharing under dichotomous preferences”. In: *Mathematical Social Sciences* 73 (2015), pp. 1–5

H. Aziz, A. Bogomolnaia, and H. Moulin. “Fair mixing: the case of dichotomous preferences”. In: *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*. 2019, pp. 753–781

A. Guerdjikova and K. Nehring. “Weighing Experts, Weighing Sources: The Diversity Value”. Working paper. 2014

Applications

- ▶ *Randomization*
 - ▶ Interpretation of probability as lotteries.
 - ▶ Use randomization for fairness.
- ▶ *Repeated decisions*
 - ▶ Alternate projects for recurring decisions.
 - ▶ Example: Mix seminar days based on polls (10% Wed, 50% Thu, 40% Fri), mix restaurants to go lunch to.
- ▶ *Budget division*
 - ▶ Decide budget division among projects via voting.
 - ▶ Non-monetary budgets are also possible: e.g., class time distribution based on student interests.
- ▶ *Approval-based apportionment*
- ▶ *Weighing criteria*
 - ▶ Organization has to make decisions in the future, based on multiple criteria. Voters say which criteria are important to them.
- ▶ *Weighing experts*
 - ▶ Each competence or perspective is a (weighted) voter approving all experts with that competence.

Axioms

- ▶ *Efficiency*: When the rule selects p , there cannot be another distribution q with $u_i(q) \geq u_i(p)$ for all $i \in N$ and $u_i(q) > u_i(p)$ for some $i \in N$.
- ▶ *Strategyproofness*
- ▶ *Monotonicity*: If a voter starts approving x and nothing else changes, then p_x weakly increases.
- ▶ *Fairness axioms*
 - ▶ *Positive share*: $u_i(p) > 0$ for all $i \in N$.
 - ▶ *Individual fair share*: $u_i(p) \geq \frac{1}{n}$ for all $i \in N$.
 - ▶ *Group fair share*: For all $S \subseteq N$, writing $A_S = \bigcup_{i \in S} A_i$, we have $\sum_{x \in A_S} p_x \geq \frac{|S|}{|N|}$.
 - ▶ *Decomposability*: We can write $p = p_1 + \dots + p_n$, where each p_i is a distribution summing to $\frac{1}{n}$ and only having support on i 's approved projects.

Theorem

A distribution p is decomposable if and only if it satisfies group fair share.

Utilitarian rule

- ▶ Select a distribution p maximizing $\sum_{i \in N} u_i(p)$.
- ▶ Equivalently, put 100% on the approval winner(s).
- ▶ For concreteness, take uniform distribution on approval winners.

✓ **efficiency** is satisfied.

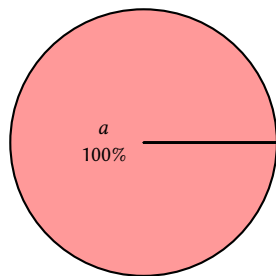
✗ **positive share** is failed.

✓ **strategyproofness** is satisfied, for the same reason that approval voting is strategyproof under dichotomous preferences.

✓ **monotonicity** is satisfied because strategyproofness implies monotonicity.

✓ **participation** is satisfied in weak versions.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Conditional utilitarian rule

- ▶ Select a distribution p maximizing $\sum_{i \in N} u_i(p)$ subject to p being decomposable.
- ▶ Equivalently, each agent $i \in N$ gets $1/n$ probability mass, and spreads it uniformly among projects that i approves and that have highest approval score.

✗ **efficiency** is failed: in the example, $0.7a + 0.3b$ is a Pareto improvement. But no decomposable distribution can dominate!

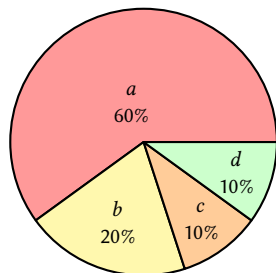
✓ **decomposability** is satisfied.

✓ **strategyproofness** is satisfied.

✓ **monotonicity** is satisfied because strategyproofness implies monotonicity.

✓ **participation** is satisfied in strong versions.

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓

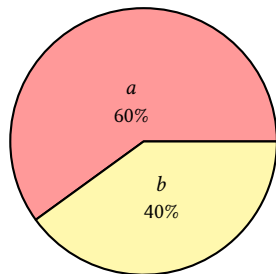


Nash rule

- ▶ Select a distribution p maximizing $\prod_{i \in N} u_i(p)$.

- ✓ **efficiency** is satisfied.
- ✓ **decomposability** is satisfied.
- ✗ **strategyproofness** is failed.
- ✗ **monotonicity** is failed.
- ✓ **participation** is satisfied in strong versions.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Nash rule: axiomatic characterization

Nash rule is the unique rule that satisfies

- ▶ convex-valuedness, continuity
- ▶ reinforcement
- ▶ ex post dominance: if a project is dominated, it gets 0.
- ▶ exclusion: if we delete an alternative that gets 0, the result does not change.
- ▶ proportionality: be decomposable on profiles where every vote is a singleton

A. Guerdjikova and K. Nehring. “Weighing Experts, Weighing Sources: The Diversity Value”. *Working paper*. 2014

Nash rule: decomposability and computation

- ▶ Nash satisfies decomposability, because it satisfies a cool **fixed point property**.
- ▶ Let p be the Nash outcome, and fix some $i \in N$. Let p_i be the distribution with

$$p_i(y) = \frac{1}{n} \cdot \frac{p_y}{\sum_{x \in A_i} p_x} \quad \text{for all } y \in A_i, \text{ and } 0 \text{ otherwise.}$$

- ▶ Then $p = p_1 + \dots + p_n$.
- ▶ This suggests a “**proportional response dynamic**” for computing Nash (start with uniform distribution, then iterate). This converges (quite fast in practice).
- ▶ Nash is equivalent to **Lindahl equilibrium** from the theory of public goods.

A. Guerdjikova and K. Nehring. “Weighing Experts, Weighing Sources: The Diversity Value”. Working paper. 2014

T. Cover. “An algorithm for maximizing expected log investment return”. In: *IEEE Transactions on Information Theory* 30.2 (1984), pp. 369–373

B. Fain, A. Goel, and K. Munagala. “The core of the participatory budgeting problem”. In: *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*. Lecture Notes in Computer Science (LNCS). Springer-Verlag, 2016, pp. 384–399

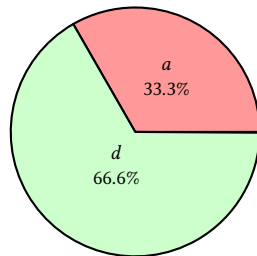
Nash rule: monotonicity

✗ **monotonicity** is failed.

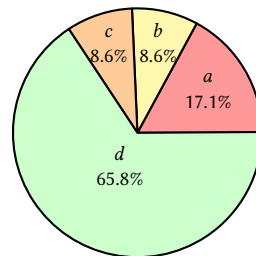
Smallest example has $m = 4$ and $n = 9$.

Have not found any examples with a “large” violation.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			
Voter 2	✓	✓		
Voter 3	✓		✓	
Voter 4		✓	✓	✓
Voter 5		✓	✓	✓
Voter 6		✓		✓
Voter 7		✓		✓
Voter 8			✓	✓
Voter 9			✓	✓



	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			✓
Voter 2	✓	✓		
Voter 3	✓		✓	
Voter 4		✓	✓	✓
Voter 5		✓	✓	✓
Voter 6		✓		✓
Voter 7		✓		✓
Voter 8			✓	✓
Voter 9			✓	✓



Axioms

	utilitarian	cond. utilitarian	Nash
efficiency	✓	–	✓
fairness	–	✓	✓
strategyproofness	✓	✓	–

Axioms

Theorem

No rule is anonymous, neutral, efficient, strategyproof, and satisfies individual fair share ($u_i(p) \geq \frac{1}{n}$) when $n \geq 5$ and $m \geq 17$.

A. Bogomolnaia, H. Moulin, and R. Stong. "Collective choice under dichotomous preferences". In: *Journal of Economic Theory* 122.2 (2005), pp. 165–184

Quotes: "We submit as a challenging conjecture the following statement: there is no strategyproof and *ex ante* efficient mechanism guaranteeing **positive shares**", "we suspect the answer is negative when [the numbers of agents and projects] are **large enough**", "we have not been able to determine if one of the **anonymity or neutrality** property (or both) can be dropped."

Surprisingly simple

Theorem

No rule is anonymous, neutral, efficient, strategyproof, and satisfies positive share ($u_i(p) > 0$) when $n \geq 5$ and $m \geq 4$.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5	✓	✓		

b and *c* are symmetric, so get same share.

We must have $p_b = p_c > 0$ by positive share for Voter 4.

Hence we have $u_5(p) < 1$.

Now suppose voter 5 approves *d* instead of *a*.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓

c and *d* are symmetric, so get same share.

If $p_c = p_d = \epsilon > 0$, we can move ϵ from *c* to *a* and ϵ from *d* to *b* to get a Pareto improvement.

So $p_c = p_d = 0$, and thus $p_a + p_b = 1$.

Hence voter 5 manipulated successfully.

Automatically getting an impossibility

- ▶ Could make an LP: Generate all profiles with 5 voters and 4 alternatives, add variables encoding the distribution selected by voting rule.
- ▶ Constraints for strategyproofness and positive share: easy. But how to do efficiency?
- ▶ **Theorem:** Whether a distribution is efficient depends only on its support, and efficient supports can be found in poly time.
- ▶ So one can use binary variables to encode efficiency.
- ▶ But it doesn't scale very well. A discrete encoding would be better.

SAT solving

- ▶ Note: efficiency and positive share **only depend on support** → discrete problem.
- ▶ But what about strategyproofness?
- ▶ Idea: **Weaken** strategyproofness (→ stronger impossibility)
- ▶ Use *pessimistic* strategyproofness: Manipulation is only successful if we go from utility 0 to > 0 or from < 1 to 1.
- ▶ This depends only on support.
- ▶ Now we can use **SAT solving**.

Theorem

No rule is efficient, strategyproof, and satisfies positive share ($u_i(p) > 0$) when $n \geq 6$ and $m \geq 4$.

Proof goes through **386 profiles**.

	A_1	A_2	A_3	A_4	A_5	A_6	possible supports	dominated supports
Profile 1	b	c	ab	ac	bd	cd	$\underline{bc}, \underline{abc}, bcd$	$ad \leftarrow bc$
Profile 2	b	c	abc	ac	bd	cd	\underline{bc}, bcd	$a \leftarrow c, ab \leftarrow bc, ad \leftarrow bc$
Profile 3	b	c	bc	ac	bd	cd	\underline{bc}, bcd	$a \leftarrow c, ab \leftarrow bc, ad \leftarrow bc$
Profile 4	bc	c	bc	ac	bd	cd	cd, \underline{bc}, bcd	$a \leftarrow c, ab \leftarrow bc, ad \leftarrow bc$
Profile 5	bc	c	bc	ac	bd	acd	$cd, \underline{bc}, \underline{bcd}$	$a \leftarrow c, ab \leftarrow bc, ad \leftarrow cd$
Profile 6	bc	c	bc	ac	bd	ad	cd, acd, \underline{bcd}	$ab \leftarrow cd$
Profile 7	bc	c	bc	ac	bcd	ad	ac, \underline{cd}, acd	$b \leftarrow c, ab \leftarrow ac, bd \leftarrow cd$
Profile 8	bc	c	bc	ac	cd	ad	ac, \underline{cd}, acd	$b \leftarrow c, ab \leftarrow ac, bd \leftarrow ac$
...								
Profile 190	b	bc	ab	abc	bd	cd	$bc, \underline{bd}, \underline{bcd}$	$a \leftarrow b, ac \leftarrow bc, ad \leftarrow bc$
Profile 191	b	c	ab	abc	bd	cd	$bc, \underline{bd}, \underline{bcd}$	$a \leftarrow b, ac \leftarrow bc, ad \leftarrow bc$
Profile 1	b	c	ab	ac	bd	cd	bc, abc, \underline{bcd}	$ad \leftarrow bc$

Axioms

	utilitarian	cond. utilitarian	Nash
efficiency	✓	-	✓
fairness	-	✓	✓
monotonicity	✓	✓	-

Another impossibility?

Designing efficient rules

- ▶ Reinforcement characterization “implies” that Nash is the only decomposable rule that maximizes a separable function of voter utility.

A. Guerdjikova and K. Nehring. “Weighing Experts, Weighing Sources: The Diversity Value”. Working paper. 2014

- ▶ Among rules of the form “choose p that maximizes $\sum_{i \in N} g(u_i(p))$ ”, only $g = \log$ (i.e., Nash) satisfies group fair share. (And only $g = \text{id}$ satisfies strategyproofness.)

A. Bogomolnaia, H. Moulin, and R. Stong. “Collective choice under dichotomous preferences”. Working paper. 2002

- ▶ But how else to design an efficient rule?
- ▶ Theorem: A distribution p is Pareto efficient if and only if there are positive weights $(w_i)_{i \in N}$ such that p maximizes $\sum_{i \in N} w_i \cdot u_i(p)$.
- ▶ Idea: Given a profile, vary weights until we get a decomposable distribution. Hopefully vary the weights in a way that gives a monotonic rule.

Sequential utilitarian rule

- ▶ Note that p maximizes $\sum_{i \in N} w_i \cdot u_i(p)$ iff its support consists only of projects with maximum weighted approval score.
- ▶ Start with $w_i = 1$ for all $i \in N$.
- ▶ Repeatedly:
 - ▶ For every voter who approves a w -maximum projects, we assign $\frac{1}{n}$ to those projects, and freeze these contributions.
 - ▶ Then we continuously increase the weights of all unassigned voters until a new project becomes w -maximum.

Theorem

The sequential utilitarian rule is monotonic.

However it fails participation. Smallest known example has $m = 5$ and $n = 45$. No counterexamples for $m = 4$ and $n \leq 14$, or for $m = 5$ and $n \leq 10$.

Other relaxations of strategyproofness

- ▶ **Subset strategyproofness.** Agents are only allowed to manipulate by reporting a subset of their true approval set.
- ▶ Impossibility still holds (with anonymity and neutrality, in 1 step)
- ▶ **Superset strategyproofness.** Agents are only allowed to manipulate by reporting a superset of their true approval set.
- ▶ Nash and sequential utilitarian fail this. Unknown if there is an efficient and decomposable rule satisfying this.
- ▶ But **leximin** does satisfy it. Leximin even satisfies **excludable strategyproofness**.

H. Aziz, A. Bogomolnaia, and H. Moulin. “Fair mixing: the case of dichotomous preferences”. In: *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*. 2019, pp. 753–781

X. Bei, X. Lu, and W. Suksompong. “Truthful cake sharing”. In: *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*. 2022, pp. 4809–4817

Axioms

	util.	leximin	cond. util.	Nash	seq. util.	No Rule!
Efficiency	✓	✓	–	✓	✓	⚡
↳ Decomposable Efficiency	✓	✓	✓	✓	✓	
Decomposability (GFS)	–	–	✓	✓	✓	
↳ Positive Share	–	✓	✓	✓	✓	⚡
Strategyproofness	✓	–	✓	–	–	⚡
↳ Monotonicity	✓	–	✓	–	✓	
Contribution IC	–	–	✓	✓	–	
↳ Weak Participation	✓	✓	✓	✓	–	

Other points

- ▶ Cake sharing.
- ▶ Welfare loss due to fairness: Nash and CUT obtain at least a $\frac{2}{\sqrt{m}}$ fraction of optimum utilitarian welfare.
- ▶ Linear utilities, rankings.

M. Michorzewski, D. Peters, and P. Skowron. “Price of Fairness in Budget Division and Probabilistic Social Choice”. In: *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*. Forthcoming. 2020

S. Airiau, H. Aziz, I. Caragiannis, J. Kruger, J. Lang, and D. Peters. “Portioning using Ordinal Preferences: Fairness and Efficiency”. In: *Artificial Intelligence* 314 (2023), p. 103809

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