Distribution Rules Under Dichotomous Preferences

Florian Brandl   Felix Brandt   Dominik Peters   Christian Stricker

2023-10-10

Conference on Voting Theory and Preference Aggregation
Celebrating Klaus Nehring’s 65th Birthday
Based on a paper presented at the ACM EC Conference 2021
## Distribution rules

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>Voter 1</td>
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<td>Voter 2</td>
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<td>Voter 3</td>
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<td>Voter 4</td>
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<tr>
<td>Voter 5</td>
<td>✓</td>
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</tbody>
</table>

![Pie chart](chart.png)

- **a**: 60%
- **b**: 20%
- **c**: 10%
- **d**: 10%
Model

- Set of voters, $N = \{1, \ldots, n\}$.
- Set of projects $A = \{x_1, \ldots, x_m\}$.
- Possible outcomes $\Delta(A) = \{p : A \rightarrow [0, 1] : \sum_{x \in A} p_x = 1\}$.
- Each voter $i \in N$ approves projects $A_i \subseteq A$.
- Voter gets utility $u_i(p) = \sum_{x \in A_i} p_x$ from distribution $p$.
- Voting rule takes the approval sets and outputs a distribution.


C. Duddy. “Fair sharing under dichotomous preferences”. In: Mathematical Social Sciences 73 (2015), pp. 1–5


Applications

▶ **Randomization**
  ▶ Interpretation of probability as lotteries.
  ▶ Use randomization for fairness.

▶ **Repeated decisions**
  ▶ Alternate projects for recurring decisions.
  ▶ Example: Mix seminar days based on polls (10% Wed, 50% Thu, 40% Fri), mix restaurants to go lunch to.

▶ **Budget division**
  ▶ Decide budget division among projects via voting.
  ▶ Non-monetary budgets are also possible: e.g., class time distribution based on student interests.

▶ **Approval-based apportionment**

▶ **Weighing criteria**
  ▶ Organization has to make decisions in the future, based on multiple criteria. Voters say which criteria are important to them.

▶ **Weighing experts**
  ▶ Each competence or perspective is a (weighted) voter approving all experts with that competence.
Axioms

- **Efficiency**: When the rule selects \( p \), there cannot be another distribution \( q \) with \( u_i(q) \geq u_i(p) \) for all \( i \in N \) and \( u_i(q) > u_i(p) \) for some \( i \in N \).

- **Strategyproofness**

- **Monotonicity**: If a voter starts approving \( x \) and nothing else changes, then \( p_x \) weakly increases.

- **Fairness axioms**
  - **Positive share**: \( u_i(p) > 0 \) for all \( i \in N \).
  - **Individual fair share**: \( u_i(p) \geq \frac{1}{n} \) for all \( i \in N \).
  - **Group fair share**: For all \( S \subseteq N \), writing \( A_S = \bigcup_{i \in S} A_i \), we have \( \sum_{x \in A_S} p_x \geq \frac{|S|}{|N|} \).
  - **Decomposability**: We can write \( p = p_1 + \cdots + p_n \), where each \( p_i \) is a distribution summing to \( \frac{1}{n} \) and only having support on \( i \)'s approved projects.

**Theorem**

A distribution \( p \) is decomposable if and only if it satisfies group fair share.
Utilitarian rule

- Select a distribution $p$ maximizing $\sum_{i \in N} u_i(p)$.
- Equivalently, put 100% on the approval winner(s).
- For concreteness, take uniform distribution on approval winners.

✓ **efficiency** is satisfied.
X **positive share** is failed.
✓ **strategyproofness** is satisfied, for the same reason that approval voting is strategyproof under dichotomous preferences.
✓ **monotonicity** is satisfied because strategyproofness implies monotonicity.
✓ **participation** is satisfied in weak versions.
Conditional utilitarian rule

- Select a distribution \( p \) maximizing \( \sum_{i \in N} u_i(p) \) subject to \( p \) being decomposable.

- Equivalently, each agent \( i \in N \) gets \( 1/n \) probability mass, and spreads it uniformly among projects that \( i \) approves and that have highest approval score.

\( \times \text{efficiency} \) is failed: in the example, \( 0.7a + 0.3b \) is a Pareto improvement. But no decomposable distribution can dominate!

\( \checkmark \text{decomposability} \) is satisfied.

\( \checkmark \text{strategyproofness} \) is satisfied.

\( \checkmark \text{monotonicity} \) is satisfied because strategyproofness implies monotonicity.

\( \checkmark \text{participation} \) is satisfied in strong versions.
Nash rule

- Select a distribution $p$ maximizing $\prod_{i \in N} u_i(p)$.

- $\checkmark$ **efficiency** is satisfied.
- $\checkmark$ **decomposability** is satisfied.
- $\times$ **strategyproofness** is failed.
- $\times$ **monotonicity** is failed.
- $\checkmark$ **participation** is satisfied in strong versions.
Nash rule: axiomatic characterization

Nash rule is the unique rule that satisfies

- convex-valuedness, continuity
- reinforcement
- ex post dominance: if a project is dominated, it gets 0.
- exclusion: if we delete an alternative that gets 0, the result does not change.
- proportionality: be decomposable on profiles where every vote is a singleton

Nash rule: decomposability and computation

- Nash satisfies decomposability, because it satisfies a cool fixed point property.
- Let $p$ be the Nash outcome, and fix some $i \in N$. Let $p_i$ be the distribution with

$$p_i(y) = \frac{1}{n} \cdot \frac{p_y}{\sum_{x \in A_i} p_x} \quad \text{for all } y \in A_i, \text{ and 0 otherwise.}$$

- Then $p = p_1 + \cdots + p_n$.
- This suggests a “proportional response dynamic” for computing Nash (start with uniform distribution, then iterate). This converges (quite fast in practice).
- Nash is equivalent to Lindahl equilibrium from the theory of public goods.

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Nash rule: monotonicity

\( \times \textbf{monotonicity} \) is failed.

Smallest example has \( m = 4 \) and \( n = 9 \).

Have not found any examples with a “large” violation.
## Axioms

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<tr>
<td>strategyproofness</td>
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</table>
Axioms

Theorem

No rule is anonymous, neutral, efficient, strategyproof, and satisfies individual fair share \( u_i(p) \geq \frac{1}{n} \) when \( n \geq 5 \) and \( m \geq 17 \).


Quotes: “We submit as a challenging conjecture the following statement: there is no strategyproof and \textit{ex ante} efficient mechanism guaranteeing positive shares”, “we suspect the answer is negative when [the numbers of agents and projects] are large enough”, “we have not been able to determine if one of the anonymity or neutrality property (or both) can be dropped.”
Surprisingly simple

**Theorem**

No rule is anonymous, neutral, efficient, strategyproof, and satisfies positive share \((u_i(p) > 0)\) when \(n \geq 5\) and \(m \geq 4\).

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\(b\) and \(c\) are symmetric, so get same share.
We must have \(p_b = p_c > 0\) by positive share for Voter 4.
Hence we have \(u_5(p) < 1\).
Now suppose voter 5 approves \(d\) instead of \(a\).

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\(c\) and \(d\) are symmetric, so get same share.
If \(p_c = p_d = \epsilon > 0\), we can move \(\epsilon\) from \(c\) to \(a\) and \(\epsilon\) from \(d\) to \(b\) to get a Pareto improvement.
So \(p_c = p_d = 0\), and thus \(p_a + p_b = 1\).
Hence voter 5 manipulated successfully.
Automatically getting an impossibility

- Could make an LP: Generate all profiles with 5 voters and 4 alternatives, add variables encoding the distribution selected by voting rule.
- Constraints for strategyproofness and positive share: easy. But how to do efficiency?
- **Theorem**: Whether a distribution is efficient depends only on its support, and efficient supports can be found in poly time.
- So one can use binary variables to encode efficiency.
- But it doesn’t scale very well. A discrete encoding would be better.
Note: efficiency and positive share only depend on support → discrete problem.

But what about strategyproofness?

Idea: Weaken strategyproofness (→ stronger impossibility)

Use pessimistic strategyproofness: Manipulation is only successful if we go from utility 0 to > 0 of from < 1 to 1.

This depends only on support.

Now we can use SAT solving.

**Theorem**

*No rule is efficient, strategyproof, and satisfies positive share (\(u_i(p) > 0\)) when \(n \geq 6\) and \(m \geq 4\).*

Proof goes through 386 profiles.
<table>
<thead>
<tr>
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<td>$cd$</td>
<td>$bc, abc, bcd$</td>
<td>$ad \leftrightarrow bc$</td>
</tr>
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<td>$abc$</td>
<td>$ac$</td>
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<td>$cd$</td>
<td>$bc, bcd$</td>
<td>$a \leftrightarrow c, ab \leftrightarrow bc, ad \leftrightarrow bc$</td>
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<td>✓</td>
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<tr>
<td>monotonicity</td>
<td>✓</td>
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Another impossibility?
Designing efficient rules

- Reinforcement characterization “implies” that Nash is the only decomposable rule that maximizes a separable function of voter utility.
  

- Among rules of the form “choose $p$ that maximizes $\sum_{i \in N} g(u_i(p))$”, only $g = \log$ (i.e., Nash) satisfies group fair share. (And only $g = id$ satisfies strategyproofness.)
  

- But how else to design an efficient rule?

- Theorem: A distribution $p$ is Pareto efficient if and only if there are positive weights $(w_i)_{i \in N}$ such that $p$ maximizes $\sum_{i \in N} w_i \cdot u_i(p)$.

- Idea: Given a profile, vary weights until we get a decomposable distribution. Hopefully vary the weights in a way that gives a monotonic rule.
Sequential utilitarian rule

Note that $p$ maximizes $\sum_{i \in N} w_i \cdot u_i(p)$ iff its support consists only of projects with maximum weighted approval score.

Start with $w_i = 1$ for all $i \in N$.

Repeatedly:

- For every voter who approves a $w$-maximum projects, we assign $\frac{1}{n}$ to those projects, and freeze these contributions.
- Then we continuously increase the weights of all unassigned voters until a new project becomes $w$-maximum.

**Theorem**

*The sequential utilitarian rule is monotonic.*

However it fails participation. Smallest known example has $m = 5$ and $n = 45$. No counterexamples for $m = 4$ and $n \leq 14$, or for $m = 5$ and $n \leq 10$. 
Other relaxations of strategyproofness

- **Subset strategyproofness.** Agents are only allowed to manipulate by reporting a subset of their true approval set.
- Impossibility still holds (with anonymity and neutrality, in 1 step)
- **Superset strategyproofness.** Agents are only allowed to manipulate by reporting a superset of their true approval set.
- Nash and sequential utilitarian fail this. Unknown if there is an efficient and decomposable rule satisfying this.
- But leximin does satisfy it. Leximin even satisfies **excludable strategyproofness.**

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## Axioms

<table>
<thead>
<tr>
<th></th>
<th>util.</th>
<th>leximin</th>
<th>cond. util.</th>
<th>Nash</th>
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<th>No Rule!</th>
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Other points

- Cake sharing.
- Welfare loss due to fairness: Nash and CUT obtain at least a $\frac{2}{\sqrt{m}}$ fraction of optimum utilitarian welfare.
- Linear utilities, rankings.

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