Recognising Multidimensional Euclidean Preferences

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Euclidean Preferences

Let $d \geqslant 1$ be an integer, let V be a finite set of voters, let A be a finite set of alternatives.

Definition of d-Euclidean preferences

A preference profile $(\succcurlyeq_i)_{i \in V}$ of linear orders is called d-**Euclidean** if there exists a map $x : V \cup A \to \mathbb{R}^d$ such that

$$a \succ_v b \iff \|x(v) - x(a)\| < \|x(v) - x(b)\|$$

for all $v \in V$ and all $a, b \in A$.

Here, $\|(x_1, \ldots, x_d)\| = \|(x_1, \ldots, x_d)\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$.

Euclidean Preferences: Direction of the Arrow

$$a \succ_v b \iff \|x(v) - x(a)\| < \|x(v) - x(b)\| \quad (1)$$

$$a \succ_v b \implies \|x(v) - x(a)\| < \|x(v) - x(b)\| \quad (2)$$

$$a \succ_v b \impliedby \|x(v) - x(a)\| < \|x(v) - x(b)\| \quad (3)$$

(This becomes more pressing when we allow ties.)

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(This becomes more pressing when we allow ties.)

(1): ties = equidistant (Bogomolnaia and Laslier 2007)
(2): my favourite; ties impose no constraints
(3): multidimensional unfolding; degeneracies

Reconition Problem

d-EUCLIDEAN	
Instance:	set A of alternatives, profile V of strict orders over A
Question:	is V d-Euclidean?

Case $d = 1$

For one dimension, the problem is solvable in polynomial time (Doignon and Falmagne 1994): use single-peakedness and single-crossingness to find the ordinal order of alternatives within $\mathbb{R}$, then use a linear program to search for the precise numbers.

Open: can you do this without solving an LP?

Case $d \geqslant 2$: this paper.

Main Result

Theorem.

For each fixed $d \geqslant 2$, the problem d-EUCLIDEAN is NP-hard. More precisely, the problem is $\exists\mathbb{R}$-complete, that is, equivalent to the *existential theory of the reals*. Thus, it is contained in PSPACE.

Theory of the Reals

Formulas of the first-order *theory of the reals* are built from

- variable symbols x_i
- constant symbols 0 and 1
- addition, subtraction, multiplication symbols
- the equality $(=)$ and inequality $(<)$ symbols
- Boolean connectives $(\vee, \wedge, \neg)$
- universal and existential quantifiers $(\forall, \exists)$

The *theory* of the reals $=$ all true sentences in this language.
(interpreted using the obvious semantics)

Existential Theory of the Reals

The *existential theory of the reals (ETR)* consists of the true sentences of the form

$$\exists x_1 \in \mathbb{R}\, \exists x_2 \in \mathbb{R} \ldots \exists x_n \in \mathbb{R} \quad F(x_1, x_2, \ldots, x_n)$$

with $F(x_1, x_2, \ldots, x_n)$ a quantifier-free formula.

In other words, F is a Boolean combination of equalities and inequalities of real polynomials.

Definition of $\exists\mathbb{R}$

L is in the complexity class $\exists\mathbb{R}$ if L is poly-time reducible to the problem of deciding whether a given sentence is in ETR (i.e., true).

d-EUCLIDEAN: Containment

d-EUCLIDEAN is contained in $\exists\mathbb{R}$ for every $d \geqslant 1$.

Proof.

A profile is d-Euclidean if and only if there **exist** reals $x_{r,i} \in \mathbb{R}$ for each $r \in A \cup V$ and $i \in [d]$ such that if $a \succcurlyeq_v b$, then

$$\sum_{i=1}^{d} (x_{v,i} - x_{a,i})^2 < \sum_{i=1}^{d} (x_{v,i} - x_{b,i})^2 .$$

Thus, the problem is equivalent to asking whether a system of polynomial inequalities has a solution. This system can be constructed in polynomial time, given the profile. □

Some $\exists\mathbb{R}$-complete problems

"can a given combinatorial object be geometrically represented?"

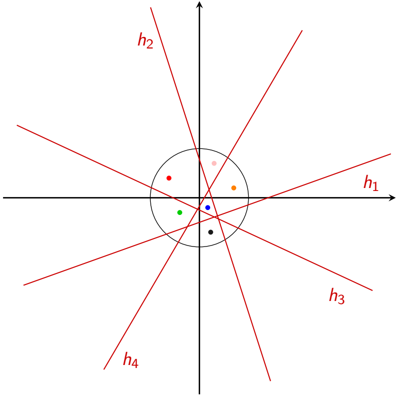
- Recognising intersection graphs of
  - line segments in the plane
  - unit disk graphs
  - unit distance graphs
  - ...
- Finding Nash equilibria in a non-cooperative game
- Realisability of hyperplane arrangements

Realisability of hyperplane arrangements

Input: a set $S \subseteq \{-,+\}^n$ of sign vectors

e.g., $S = \{(+,+,+,+), (-,+,+,-), (-,+,-,+), (-,+,-,-), (-,-,-,+), (-,-,-,-)\}$

Question: Can this be *realised* by oriented hyperplanes in $\mathbb{R}^2$?



Hardness

Theorem.

For each fixed $d \geqslant 2$, the problem d-EUCLIDEAN is $\exists\mathbb{R}$-complete.

Theorem.

Recognising d-Euclidean preferences is $\exists\mathbb{R}$-complete even for dichotomous preferences.

Theorem.

Recognising d-Dichotomous-Uniform-Euclidean (d-DUE) preferences is $\exists\mathbb{R}$-complete. *(see Elkind and Lackner 2015)*

Forbidden Subprofiles: Single-Peaked

Some domain restrictions can be characterised by a finite list of *forbidden subprofiles*.

e.g., a profile is **single-peaked** iff it does not contain any of

v_1	v_2	v_3
a	b	c
b	c	a
c	a	b

v_1	v_2	v_3
a	c	a
b	b	c
c	a	b

v_1	v_2
d	d
a	c
b	b
c	a

v_1	v_2
d	c
a	d
b	b
c	a

v_1	v_2
a	c
d	d
b	b
c	a

(Ballester and Haeringer 2011)

Forbidden Subprofiles: Single-Crossing

a profile is **single-crossing** iff it does not contain any of

v_1	v_2	v_3
a	b	c
b	c	a
c	a	b

v_1	v_2	v_3
a	b	d
b	a	a
c	d	b
d	c	c

v_1	v_2	v_3
a	c	c
b	a	b
c	d	d
d	b	a

v_1	v_2	v_3
a	d	d
b	b	c
c	a	a
d	c	b

v_1	v_2	v_3
a	c	d
b	b	a
c	a	c
d	d	b

v_1	v_2	v_3
a	a	c
b	d	d
c	c	a
d	b	b

v_1	v_2	v_3
a	c	d
b	b	b
c	d	c
d	a	a

v_1	v_2	v_3
a	b	d
b	a	a
c	d	c
d	c	b

v_1	v_2	v_3
a	a	b
b	d	a
c	c	d
d	b	c

v_1	v_2	v_3
a	b	d
b	c	b
c	a	a
d	d	c

v_1	v_2	v_3
a	b	c
b	d	b
c	a	a
d	c	d

v_1	v_2	v_3
a	a	c
b	c	a
c	d	b
d	b	d

v_1	v_2	v_3
a	b	c
b	a	d
c	d	b
d	c	a

v_1	v_2	v_3
a	c	c
b	a	b
c	d	a
d	b	d

v_1	v_2	v_3
a	a	d
b	d	b
c	c	c
d	b	a

v_1	v_2	v_3
a	b	b
b	c	d
c	a	a
d	d	c

v_1	v_2	v_3
a	a	c
b	c	b
c	d	d
d	b	a

v_1	v_2	v_3
a	e	e
b	a	b
c	b	a
d	d	c
e	c	d

v_1	v_2	v_3
a	a	b
b	c	c
c	b	e
d	e	a
e	d	d

v_1	v_2	v_3
a	b	e
b	a	b
c	d	a
d	c	c
e	e	d

v_1	v_2	v_3
a	b	c
b	c	b
c	a	a
d	e	d
e	d	e

v_1	v_2	v_3
a	b	b
b	a	a
c	c	d
d	e	e
e	d	c

v_1	v_2	v_3
a	a	b
b	b	a
c	e	d
d	d	c
e	c	e

v_1	v_2	v_3
a	a	b
b	c	c
c	b	a
d	e	e
e	d	d

v_1	v_2	v_3
a	a	c
b	c	a
c	e	d
d	d	e
e	b	b

v_1	v_2	v_3
a	a	b
b	b	a
c	e	c
d	c	e
e	d	d

v_1	v_2	v_3
a	b	b
b	a	c
c	c	a
d	e	d
e	d	e

v_1	v_2	v_3
a	b	b
b	a	a
c	c	d
d	d	c
e	f	e
f	e	f

v_1	v_2	v_3	v_4
a	b	c	c
b	a	a	b
c	c	b	a

v_1	v_2	v_3	v_4
a	a	b	b
b	b	a	a
c	d	c	d
d	c	d	c

(Bredereck, Chen, and Woeginger 2013)

Forbidden Subprofiles: d-Euclidean

Theorem

For each fixed $d \geqslant 2$, the d-Euclidean domain cannot be characterised by finitely many forbidden subprofiles.

Subject to $P \neq \exists\mathbb{R}$, this is obvious!

Forbidden Subprofiles: d-Euclidean

Theorem

For each fixed $d \geqslant 2$, the d-Euclidean domain cannot be characterised by finitely many forbidden subprofiles.

Subject to $P \neq \exists\mathbb{R}$, this is obvious! But we can prove it without assumptions via a connection to the theory of realisability of *oriented matroids*.

Precision

Theorem

For each fixed $d \geqslant 2$, there are d-Euclidean profiles with n voters and m alternatives such that every integral Euclidean embedding uses at least one coordinate that is

$$2^{2^{\Omega(n+m)}}.$$

On the other hand, every d-Euclidean profile can be realized by an integral Euclidean embedding whose coordinates are at most

$$2^{2^{O(n+m)}}.$$

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