# COMSOC Lecture 6: Fair Division - Rent Division

Dominik Peters

Based on slides of Ulle Endriss and Nisarg Shah

CNRS, LAMSADE, Université Paris Dauphine - PSL

IASD · 2025-02-12

# PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

## Share Rent

Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.

START >


## Split Fare

Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.

START >

## Assign Credit

Determine the contribution of each individual to a school project, academic paper, or business endeavor.

START >

## Divide Goods

Fairly divide jewelry, artworks, electronics, toys, furniture, financial assets, or even an

## Distribute Tasks

Divvy up household chores, work shifts, or tasks for a school project among two or

## Suggest an App

We're always looking for ideas for new apps. Have something else you'd like to

# Rent Division

*Roommates want to move into a shared apartment, and need to allocate the rooms and split the rent.*

Model:

- Set $N = \{1, \ldots, n\}$ of roommates.
- Set $R = \{1, \ldots, n\}$ of rooms.
- The total rent to pay is 1.
- Each player $i \in N$ has a value $v_{i,r} \in \mathbb{R}$ for each room $r \in R$.
- The ("quasilinear") utility that $i$ obtains when getting room $r$ and having to pay rent $p_r$ is $v_{i,r} - p_r$.

Aim:

- A room assignment is a permutation $\sigma : N \to R$.
- An allocation $(\sigma, p)$ consists of $\sigma$ and a payment vector $p = (p_r)_{r \in R}$ with $\sum_{r \in R} p_r = 1$ (usually non-negative).
- Allocation $(\sigma, p)$ is envy-free if $v_{i,\sigma(i)} - p_{\sigma(i)} \geqslant v_{i,r} - p_r$ for all $i \in N$ and $r \in R$.

# Existence of Envy-Free Allocations

**Theorem**

*For all valuations $(v_{i,r})_{i \in N, r \in R}$, an envy-free allocation exists.*

> L.-G. Svensson. "Large indivisibles: An analysis with respect to price equilibrium and fairness". In: *Econometrica* 51.4 (1983), pp. 939–954

*Proof strategy*: Take any optimal room assignment $\sigma$, prove using linear programming duality that there exist prices $p$ such that $(\sigma, p)$ is envy-free.

A room assignment $\sigma$ is optimal if it maximizes $\sum_{i \in N} v_{i,\sigma(i)}$.

# Existence of Envy-Free Allocations: Proof

Let $\sigma$ be an optimal room assignment. Standard result from the theory of bipartite matching: The linear program

$$\max \sum_{i \in N} \sum_{r \in R} v_{i,r} x_{i,r} \text{ s.t. } \begin{aligned} &\sum_{i \in N} x_{i,r} = 1 \text{ for } r \in R \\ &\sum_{r \in r} x_{i,r} = 1 \text{ for } i \in N \\ &x_{i,r} \geqslant 0 \end{aligned}$$

has an optimal solution with $x_{i,r} = 1$ when $r = \sigma(i)$. Thus the dual

$$\min \sum_{i \in N} q_i + \sum_{r \in R} p_r \text{ s.t. } q_i + p_r \geqslant v_{ir} \text{ for } i \in N,\ r \in R.$$

must have an optimum solution where constraint holds with equality whenever $r = \sigma(i)$. Hence $(p_r)_{r \in R}$ is an envy-free price vector (though we may need to shift it to get it to sum to 1). $\square$

# Properties of Envy-Free Allocations: Lemma 1

**Lemma**

*If $(\sigma, p)$ is envy-free then $\sigma$ is optimal.*

**Proof.**

Let $\pi$ be any other room assignment. From envy-freeness, $v_{i,\sigma(i)} - p_{\sigma(i)} \geqslant v_{i,\pi(i)} - p_{\pi(i)}$ for all $i \in N$ and $r \in R$.

Summing these over all $i \in N$,

$$\sum_{i \in N} v_{i,\sigma(i)} - \underbrace{\sum_{i \in N} p_{\sigma(i)}}_{=1} \geqslant \sum_{i \in N} v_{i,\pi(i)} - \underbrace{\sum_{i \in N} p_{\pi(i)}}_{=1} .$$

Thus, $\sum_{i \in N} v_{i,\sigma(i)} \geqslant \sum_{i \in N} v_{i,\pi(i)}$. So $\sigma$ is optimal. □

# Properties of Envy-Free Allocations: Lemma 2

**Lemma**

*Let $\sigma_1$, $\sigma_2$ be optimal room assignments, and let $(\sigma_1, p)$ be an envy-free allocation. Then $(\sigma_2, p)$ is also an envy-free allocation.*

**Proof.**

Exercise. □

# Properties of Envy-Free Allocations: Lemma 2

**Lemma**

*Let $\sigma_1$, $\sigma_2$ be optimal room assignments, and let $(\sigma_1, p)$ be an envy-free allocation. Then $(\sigma_2, p)$ is also an envy-free allocation, with all agents indifferent between the two: $v_{i,\sigma_1(i)} - p_{\sigma_1(i)} = v_{i,\sigma_2(i)} - p_{\sigma_2(i)}$ for all $i \in N$.*

**Proof.**

We show $v_{i,\sigma_1(i)} - p_{\sigma_1(i)} = v_{i,\sigma_2(i)} - p_{\sigma_2(i)}$ for $i \in N$. From this, envy-freeness of $(\sigma_2, p)$ follows immediately.

We have $v_{i,\sigma_1(i)} - p_{\sigma_1(i)} \geqslant v_{i,\sigma_2(i)} - p_{\sigma_2(i)}$ for all $i \in N$ since $(\sigma_1, p)$ is envy-free. Sum these inequalities to get

$$\left(\sum_{i \in N} v_{i,\sigma_1(i)}\right) - 1 \geqslant \left(\sum_{i \in N} v_{i,\sigma_2(i)}\right) - 1.$$

But the two sides of this inequality are equal, since both $\sigma_1$ and $\sigma_2$ are optimal. Hence each inequality is satisfied with equality, as required. □

# Computing an Envy-Free Allocations

Algorithm:

1. Find an optimal room assignment $\sigma$ using maximum weighted bipartite matching.
2. Compute prices $p$ that make $(\sigma, p)$ envy-free, using a linear program.

This algorithm is correct:

- Let $\sigma$ be the room assignment found in Step 1.
- From the existence theorem, there exists some envy-free allocation $(\pi, p)$.
- From Lemma 1, $\pi$ is optimal.
- Since both $\pi$ and $\sigma$ are optimal, Lemma 2 implies that $(\sigma, p)$ is envy-free.

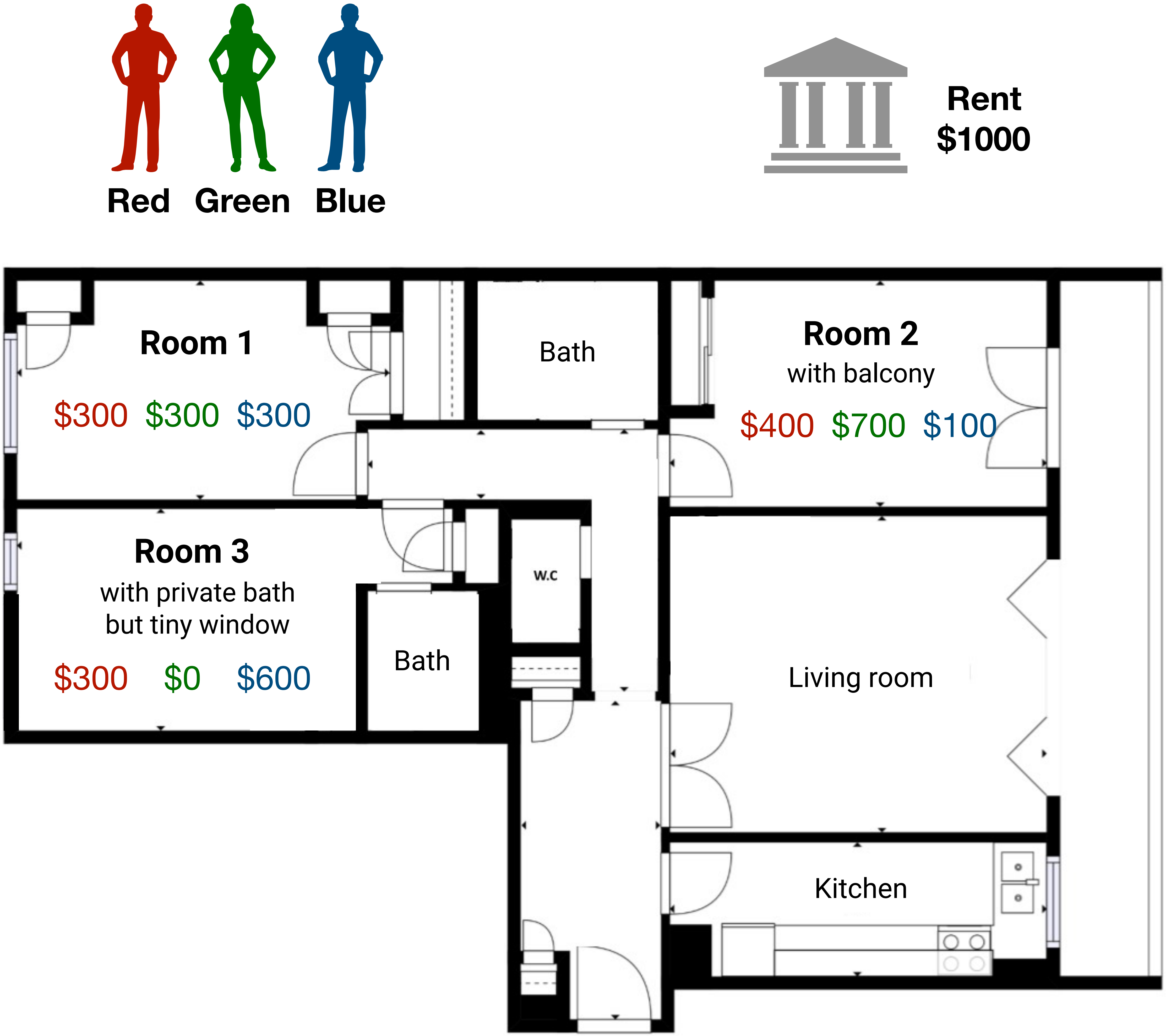
# Selecting an Envy-Free Allocations

There can be many different envy-free allocations. How to decide which one to use?

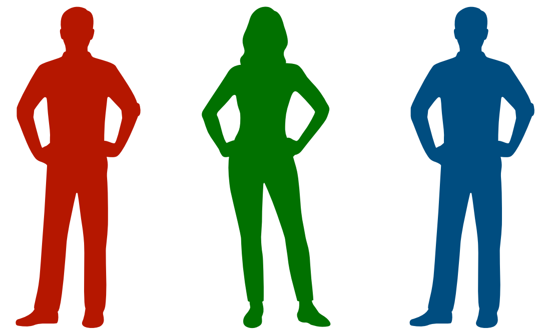
Spliddit.org uses the maximin method: subject to envy-freeness, maximize the minimum utility (“egalitarian welfare”).

$$\max L \;\text{ s.t. }\; \begin{aligned} & v_{i,\sigma(i)} - p_{\sigma(i)} \geqslant L \;\text{ for } i \in N \\ & v_{i,\sigma(i)} - p_{\sigma(i)} \geqslant v_{i,r} - p_r \;\text{ for } i \in N \text{ and } r \in R \\ & \sum_{r \in R} p_r = 1. \end{aligned}$$

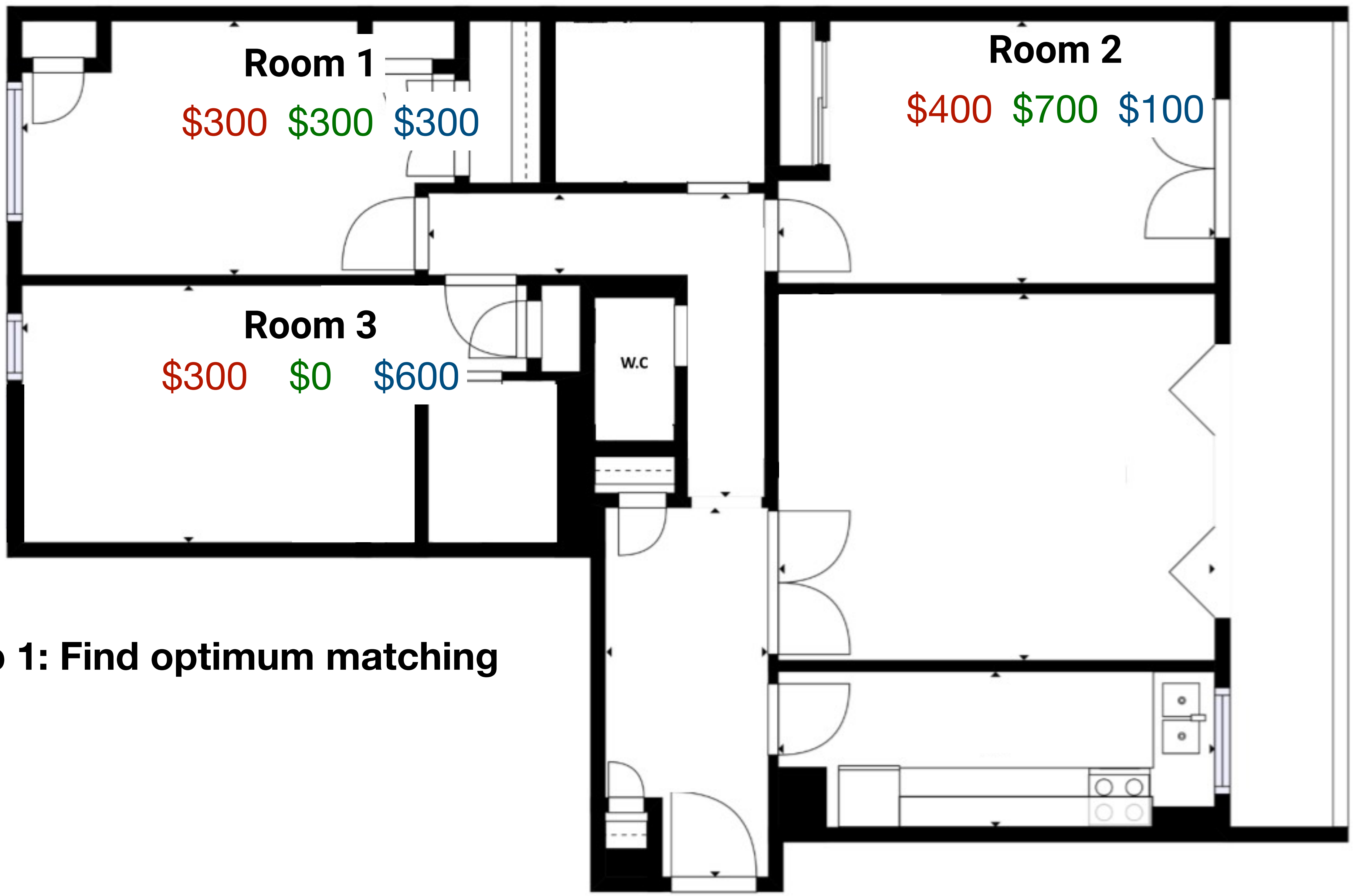
Y. Gal et al. “Which is the fairest (rent division) of them all?” In: *Journal of the ACM* 64.6 (2017), Article 39



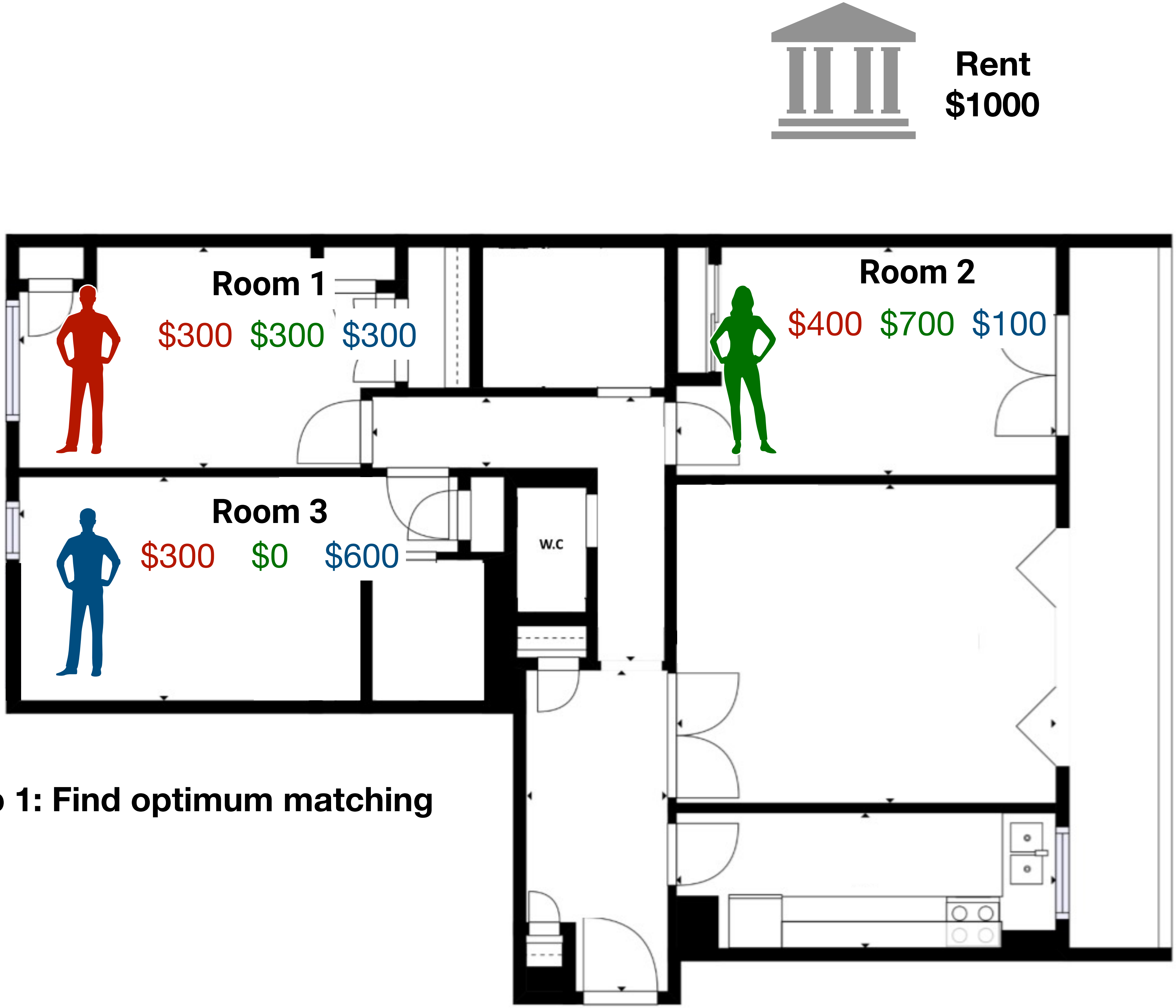
Red
Green
Blue
Rent
$1000
Room 1
$300 $300 $300
Bath
Room 2
with balcony
$400 $700 $100
Room 3
with private bath
but tiny window
$300 $0 $600
Bath
W.C
Living room
Kitchen



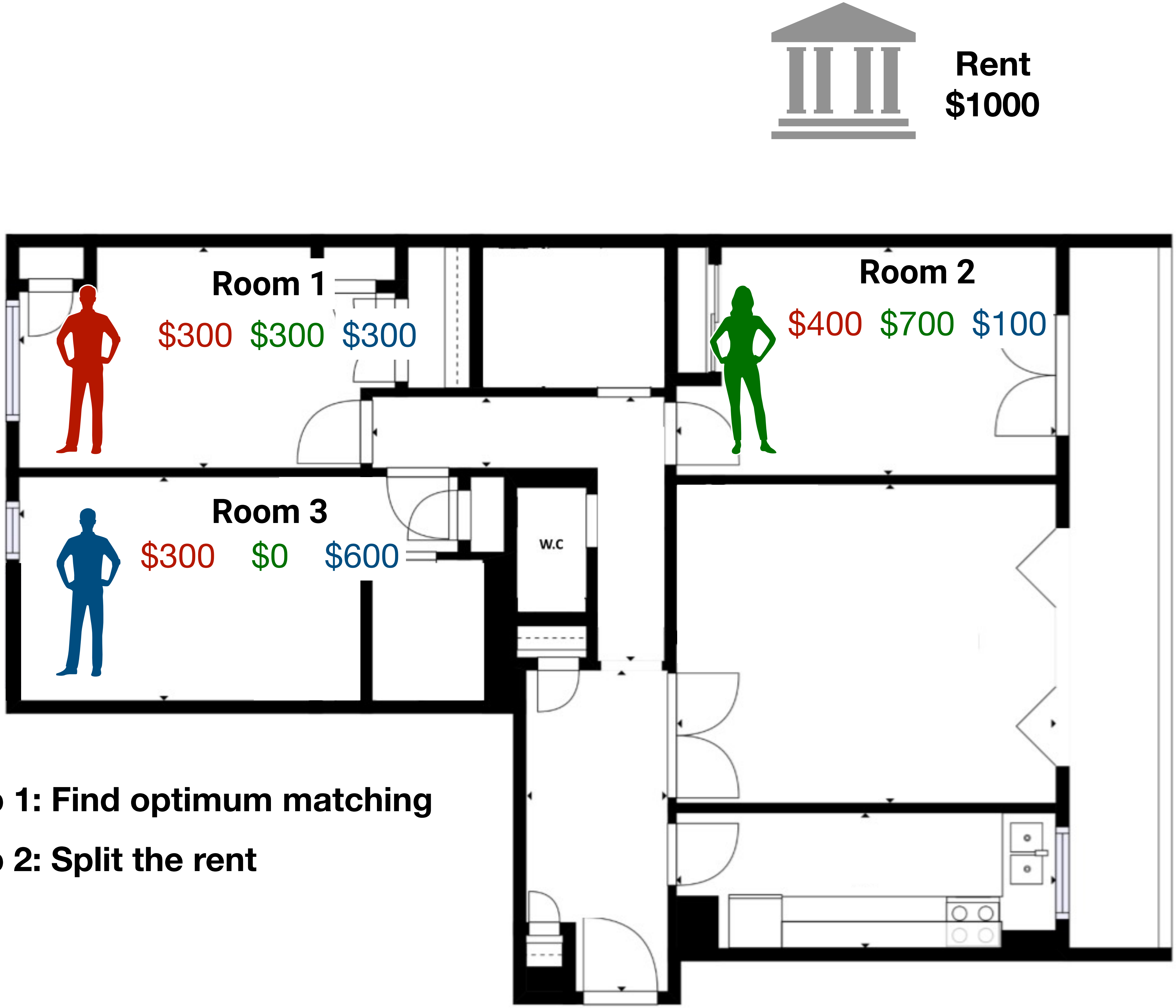

Rent
$1000




**Step 1: Find optimum matching**

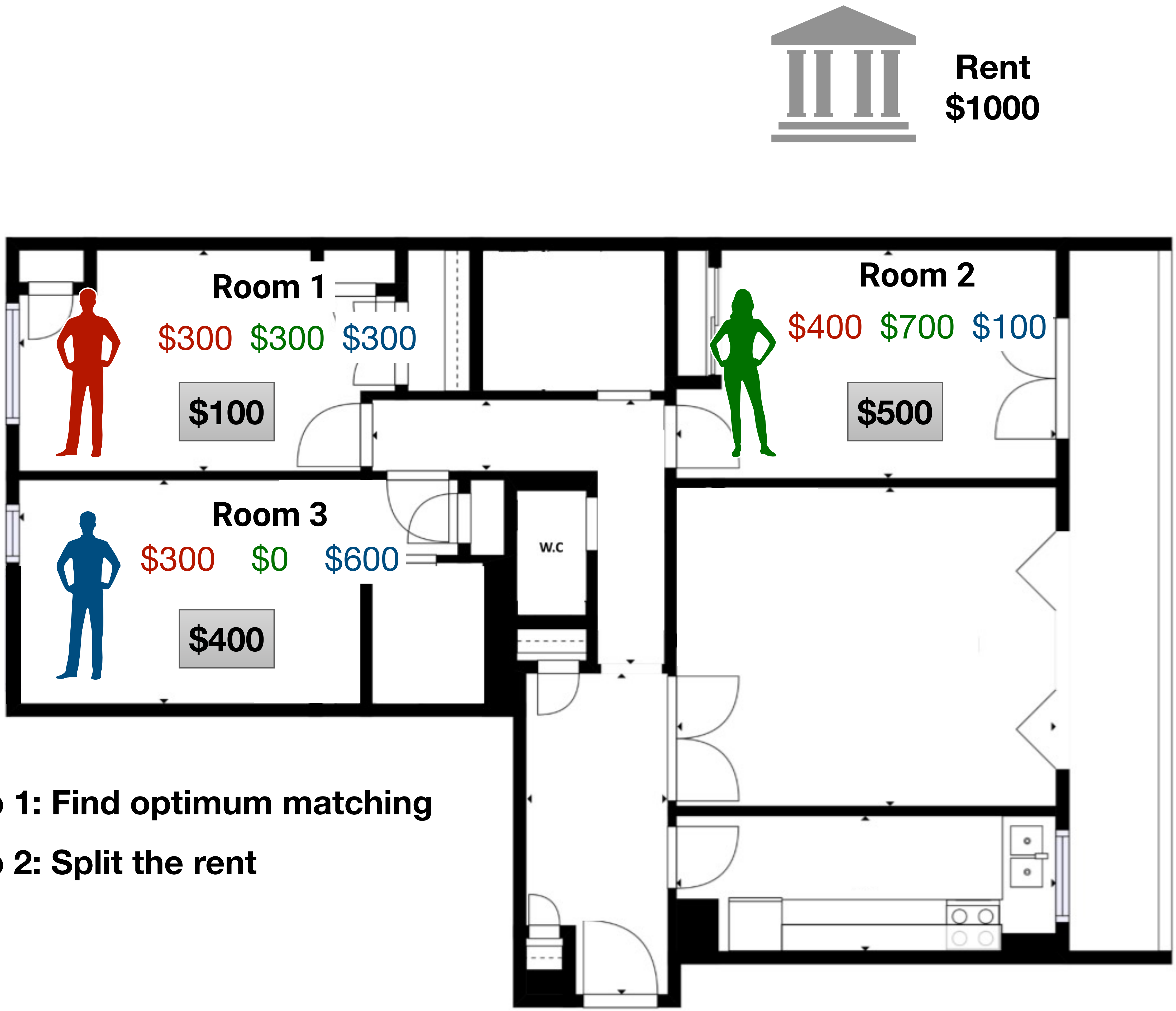



**Step 1: Find optimum matching**



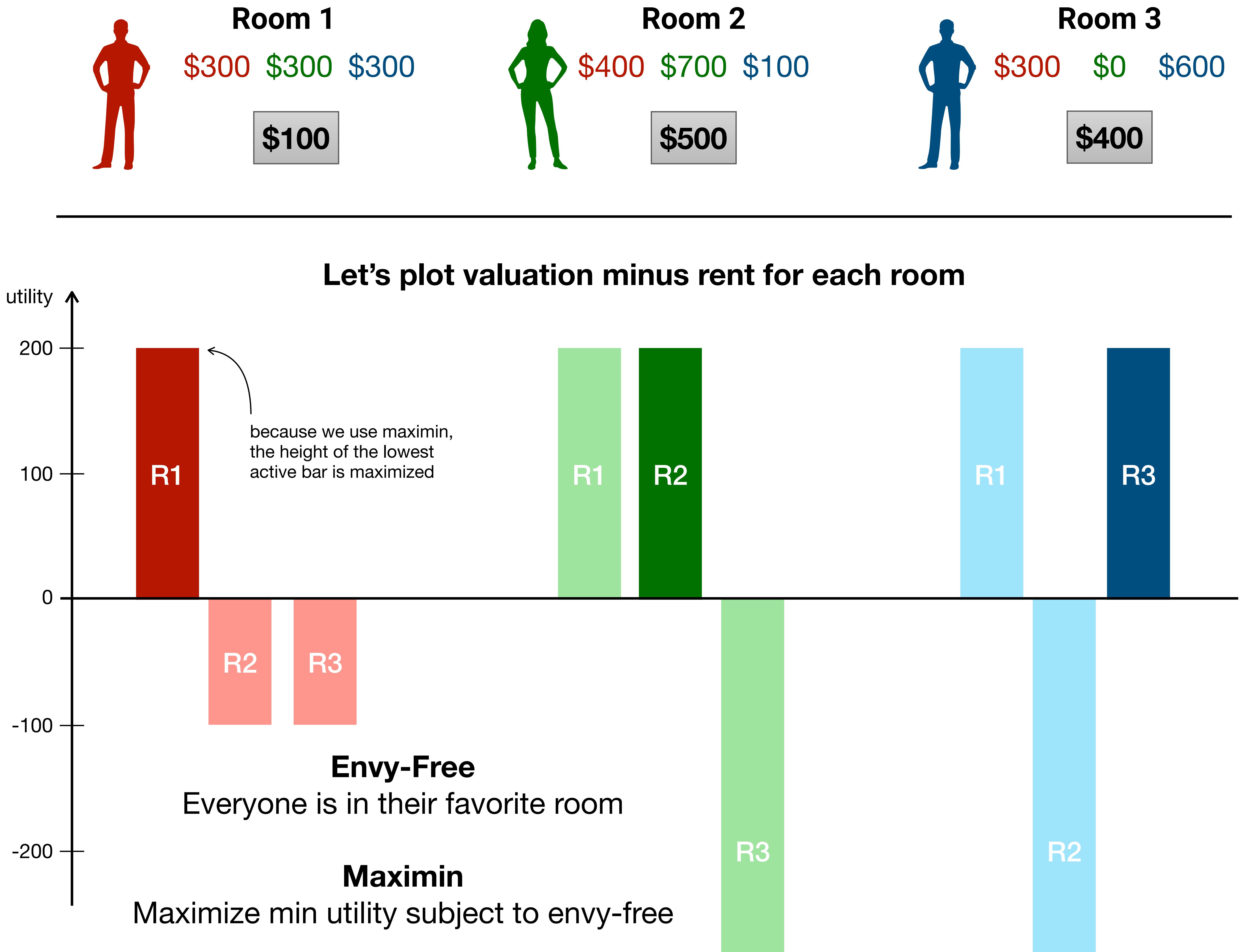

**Step 1: Find optimum matching**

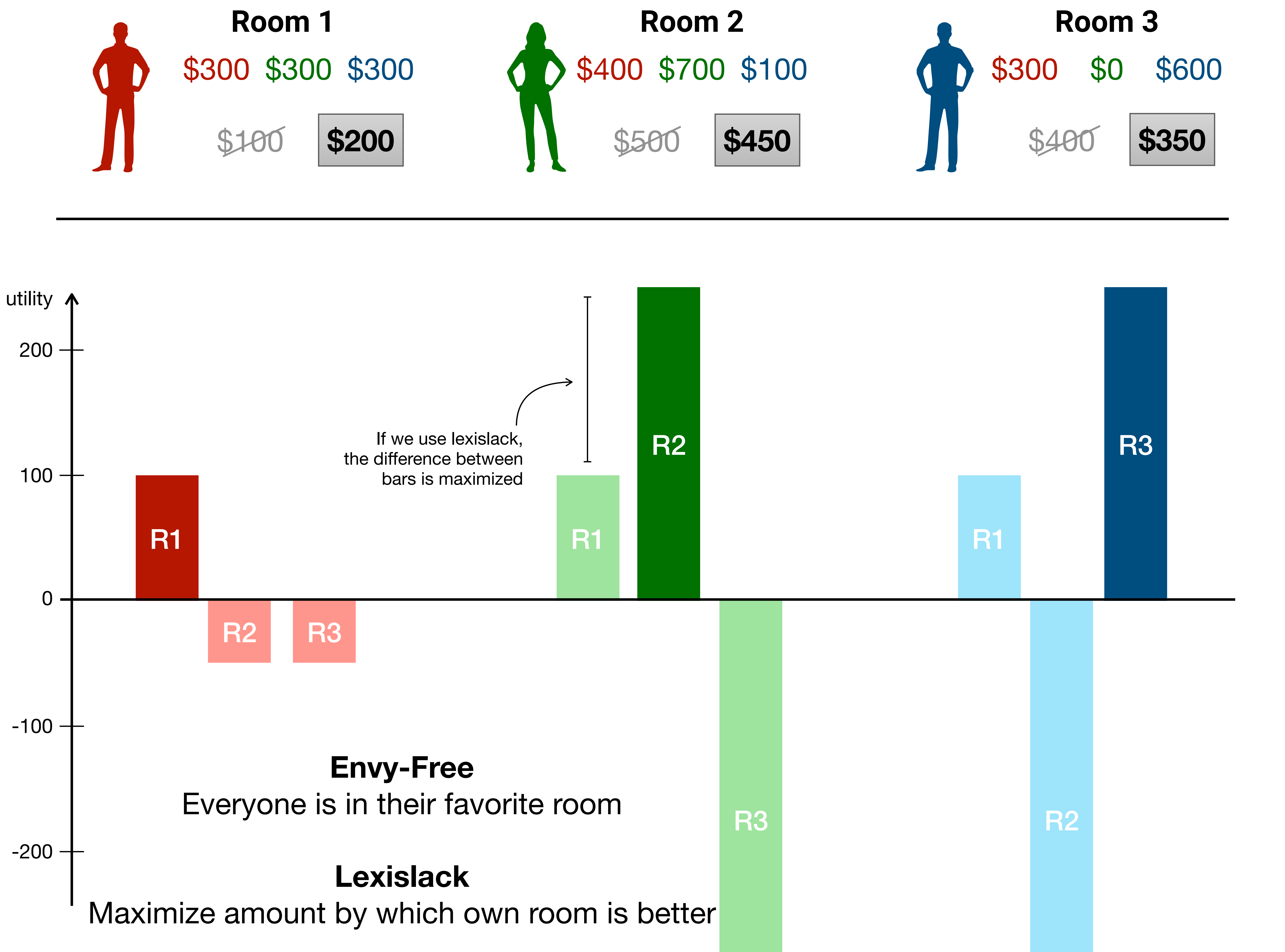
**Step 2: Split the rent**




**Step 1: Find optimum matching**

**Step 2: Split the rent**


Room 1
$300 $300 $300
$100
Room 2
$400 $700 $100
$500
Room 3
$300 $0 $600
$400
Let's plot valuation minus rent for each room
utility
200
100
0
-100
-200
R1
R2
R3
R1
R2
R3
R1
R3
R2
because we use maximin,
the height of the lowest
active bar is maximized
Envy-Free
Everyone is in their favorite room
Maximin
Maximize min utility subject to envy-free



Room 1
$300 $300 $300
~~$100~~ $200
Room 2
$400 $700 $100
~~$500~~ $450
Room 3
$300 $0 $600
~~$400~~ $350
utility
200
100
0
-100
-200
R1
R2
R3
R1
R2
R3
R1
R2
R3
If we use lexislack, the difference between bars is maximized
Envy-Free
Everyone is in their favorite room
Lexislack
Maximize amount by which own room is better

# The Lexislack Rule

- It ensures envy-freeness in a maximum-size L1-ball around the valuations
- It can be computed in polynomial time by linear programming
- It is substantially (20%) more robust than the maximin rule, when evaluated on Spliddit data
- Try it out: https://pref.tools/rent

# Lexislack

Maxislack rule:

$$\begin{aligned} \text{maximize} \quad & L \\ \text{subject to} \quad & (v_{i\sigma(i)} - p_{\sigma(i)}) - (v_{i\sigma(j)} - p_{\sigma(j)}) \geqslant L \quad \forall i \neq j \\ & \textstyle\sum_{r \in R} p_r = 1 \\ & p_r \in \mathbb{R} \quad \forall r \in R \end{aligned}$$

Better to use leximax, giving the *lexislack rule*.

- The lexislack rule is essentially single-valued.
- A lexislack allocation can be found in polynomial time by solving $O(n^4)$ linear programs.

# Probabilistic approach

A distribution $\mathcal{D}$ over valuations $v$ is obtained (for example) by asking agents for valuations, and then adding noise (e.g., Gaussian or uniform) around those valuations. Our goal will be to find an allocation $(\sigma, p)$ that maximizes the probability of being envy-free with respect to $\mathcal{D}$:

$$\text{EFrate}_{\mathcal{D}}(\sigma, p) = \mathbb{P}_{v \sim \mathcal{D}}[(\sigma, p) \text{ is envy-free under } v].$$

How to solve this optimization? Unclear. Our approach: sample from distribution, find sample optimal allocation.

# Probabilistic approach

$$\mathrm{EFrate}_{\mathcal{D}}(\sigma, p) = \mathbb{P}_{v \sim \mathcal{D}}[(\sigma, p) \text{ is envy-free under } v].$$

**Theorem**

*Let $\varepsilon, \delta > 0$. There is a value $m \in \mathbb{N}$ with*

$$m = O\left(\frac{n^2 \log n + \log(1/\delta)}{\varepsilon^2}\right)$$

*such that for every probability distribution $\mathcal{D}$ over valuation profiles, if $S$ is a collection of at least $m$ samples drawn i.i.d. from $\mathcal{D}$, and $(\sigma^*, p^*)$ is the allocation that maximizes $\mathrm{EFrate}_S$, then with probability at least $1 - \delta$,*

$$\mathrm{EFrate}_{\mathcal{D}}(\sigma^*, p^*) \geqslant \max_{(\sigma,p)} \mathrm{EFrate}_{\mathcal{D}}(\sigma, p) - \varepsilon.$$

# Complexity

EF-RATE MAXIMIZATION
**Input:** Set $N$ of agents, set $R$ of rooms, a list of $m$ valuation profiles $v^{(1)}, \ldots, v^{(m)}$, number $B$.
**Question**: Does there exist an allocation that is envy-free for at least $B$ of the $m$ valuation profiles?
NP-complete (reduction from Clique).

# Dividing Homogeneous Goods

Another simple fair division model:

- Set $N = \{1, \ldots, n\}$ of players.
- Set $O = \{o_1, \ldots, o_m\}$ of objects.
- Each player $i \in N$ has a value $v_{i,o} \geqslant 0$ for each object $o \in O$, for convenience normalizing so that $\sum_{o \in O} v_{i,o} = 1$.
- A (fractional) allocation is given by $(x_{i,o})_{i \in N, o \in O} \geqslant 0$ with $\sum_{i \in N} x_{i,o} = 1$ for all $o \in O$.

An allocation $(x_{i,o})$ is:

- Proportional if $\sum_{o \in O} v_{i,o} x_{i,o} \geqslant \frac{1}{n}$ for all $i \in N$
- Envy-free if $\sum_{o \in O} v_{i,o} x_{i,o} \geqslant \sum_{o \in O} v_{i,o} x_{j,o}$ for all $i, j \in N$.
- Pareto-optimal if there is no other allocation $(y_{i,o})$ with $\sum_{o \in O} v_{i,o} x_{i,o} \leqslant \sum_{o \in O} v_{i,o}, y_{i,o}$ for all $i \in N$, and the inequality is strict for at least one player.

# Dividing Homogeneous Goods: Example Rules

- Trivial uniform rule: $x_{i,o} = \frac{1}{n}$ for all $i$ and all $o$.
  Satisfies proportionality, envy-freeness, strategyproofness, but fails Pareto-optimality.
- Use a cake cutting protocol (by splitting the cake into $m$ regions) to achieve proportionality and perhaps envy-freeness. Will fail strategyproofness and Pareto.
- Serial dictatorship: Give all objects fully to player 1, except for objects that player 1 values at 0; those objects go to player 2, except for those which player 2 values at 0; those go to player 3, etc.
  Satisfies Pareto-optimality and strategyproofness.
- The Nash product rule also known as Competitive Equilibrium from Equal Income: take the allocation maximizing $\prod_{i \in N} \left( \sum_{o \in O} v_{i,o} x_{i,o} \right)$. Satisfies Pareto-optimality, proportionality, and envy-freeness.