COMSOC Lecture 4: Apportionment

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Recap

- Lecture 1: Cake cutting (protocols for proportionality and envy-freeness). Rent division (envy-freeness, maximizing minimum utility).
- Lecture 2a: Allocation of indivisible items.
- Lecture 2b: Random assignment
- Lecture 3: Matching. Resident-hospital problem, house swapping.

Exercise

Suppose we have a country consisting of 3 states. The country has a parliament with 100 seats, and needs to decide how many seats go to representatives of each state. It wants to do this proportional to the population. The populations of the 3 states are:

State A	4 400
State B	45 300
State C	50 300

Alexander Hamilton (1757–1804) proposed this apportionment method: Determine each state's fair share (the quota) by dividing its population by the total population and multiplying by the number of available seats. Then either round the quota up or down so that (1) the correct number of seats is allocated (2)we round up the states with the highest fractional part of their quota. (The fractional part of 3.141 is 0.141.)

Exercise: How would Hamilton's method apportion the seats?

Exercise

Answer:

	population	quota	rounded
State A	4 400	4.4	5
State B	45 300	45.3	45
State C	50 300	50.3	50

Exercise 2: What would happen if the legislature changed its mind and allocates 101 seats to the 3 states?

Exercise

Answer 1:

	population	quota	rounded
State A	4 400	4.4	5
State B	45 300	45.3	45
State C	50 300	50.3	50

Answer 2:

	population	quota	rounded
State A	4 400	4.444	4
State B	45 300	45.753	46
State C	50 300	50.803	51

Exercise

Answer 1:

	population	quota	rounded
State A	4 400	4.4	5
State B	45 300	45.3	45
State C	50 300	50.3	50

Exercise 3: Suppose a new census gives new population numbers:

	population
State A	4 500
State B	45 200
State C	49 000

How does this affect the apportionment of 100 seats by Hamilton's method?

Exercise

Answer 1:

	population	quota	rounded
State A	4 400	4.4	5
State B	45 300	45.3	45
State C	50 300	50.3	50

Answer 3:

	population	quota	rounded
State A	4 500	4.5	5
State B	45 200	45.2	46
State C	49 000	49.0	49

Exercise

Answer 1:

	population	quota	rounded
State A	4 400	4.4	5
State B	45 300	45.3	45
State C	50 300	50.3	50

Exercise 4: Suppose a new state is to join the union, and we increase the parliament size to 102 seats. What will be the apportionment?

	population
State A	4 400
State B	45 300
State C	50 300
State D	1 700

Exercise

Answer 1:

	population	quota	rounded
State A	4 400	4.4	5
State B	45 300	45.3	45
State C	50 300	50.3	50

Answer 4:

	population	quota	rounded
State A	4 400	4.41	4
State B	45 300	45.43	45
State C	50 300	50.45	51
State D	1 700	1.71	2

The Apportionment Problem

There are n states with populations $p_1, \ldots, p_n$. House size h.
We write this as $[h; p_1, \ldots, p_n]$. Example: $[100; 4400, 45300, 50300]$

Task: find seat counts $s_1, \ldots, s_n \in \mathbb{N}$ with $s_1 + \cdots + s_n = h$.

An apportionment method takes as input a problem $[h; p_1, \ldots, p_n]$ and outputs seat counts.[1]

[1] In this lecture, we will be sloppy about what happens in case there are ties, for example if two states have the same seat count.

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An apportionment method takes as input a problem $[h; p_1, \ldots, p_n]$ and outputs seat counts.[1]

For any $\alpha > 0$, we write $[h; p_1, \ldots, p_n] \equiv [h; \alpha p_1, \ldots, \alpha p_n]$ and say that these two problems are equivalent. For example

$$[100; 4400, 45300, 50300] \equiv [100; 4.4, 45.3, 50.3],$$

taking $\alpha = 1/1000$. An apportionment method is scale-invariant if it outputs the same seat counts for equivalent inputs.

Note that always $[h; p_1, \ldots, p_n] \equiv [h; q_1, \ldots, q_n]$ where $q_i = h \cdot p_i / \sum_{j=1}^{n} p_j$ is the quota of state i.

[1] In this lecture, we will be sloppy about what happens in case there are ties, for example if two states have the same seat count.

The Apportionment Problem: Applications

- Apportion parliament seats to states by population. Done in several countries, historically most noteworthily in the United States House of Representatives.
- Apportion parliament seats to parties by vote count. Used in several countries (notably in Europe) that use proportional voting systems. If a party gets $\alpha\%$ of the votes, then it should get approximately $\alpha\%$ of the seats.
- Allocation of identical items. Suppose there is a collection of many identical items, and we need to allocate them to n agents, where each agent has a claim on the items of different strength. Examples:
  - A transit system needs to assign trains or train drivers to metro lines, in proportion to the number of passengers on the line.
  - A school system assigning teachers to schools by their number of students.
- Rounding percentages to integers summing to 100%.

Naïve rounding does not work

We might naïvely try to just round the quota q_i of each state in the usual way (for example $3.7 \to 4$ and $3.1 \to 3$). But that does not work.

Consider the problem $[1; 1, 1, 1]$.
In quota form, this is equivalent to $[1; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$.
Rounding each quota, we get the seat assignment $(0, 0, 0)$ which does not sum to 1.

Method of Largest Remainder

Hamilton's method is also known as Method of Largest Remainder.

Write a given problem $[h; p_1, \ldots, p_n] \equiv [h; q_1, \ldots, q_n]$ in quota form.

The method works like this:

1. To each state i, assign $\lfloor q_i \rfloor$ seats.
2. The remaining number of seats is $t = h - \lfloor q_1 \rfloor - \cdots - \lfloor q_n \rfloor$. Take the t states with the largest remainder $q_i - \lfloor q_i \rfloor$, and assign each of them an extra seat.

Recall that $\lfloor q_i \rfloor$ is the largest integer that is at most q_i.
For example $\lfloor 3.14 \rfloor = 3$ and $\lfloor 10 \rfloor = 10$.

Method of Largest Remainder satisfies lower and upper quota

Write a problem $[h; p_1, \ldots, p_n] \equiv [h; q_1, \ldots, q_n]$ in quota form.

An apportionment method outputting $s_1, \ldots, s_n$ satisfies

1. lower quota if $s_i \geqslant \lfloor q_i \rfloor$ for each state i,
2. upper quota if $s_i \leqslant \lceil q_i \rceil$ for each state i.

A method that satisfies both lower quota and upper quota is called a quota method.

Theorem. The Method of Largest Remainder is a quota method.

Method of Largest Remainder: Example

state	population	quota	lower q.	upper q.	remainder	seats
1	264	2.64	2	3	0.64	3
2	361	3.61	3	4	0.61	3
3	375	3.75	3	4	0.75	4
total	1 000	10	8	11	2	10

Method of Largest Remainder gets "closest" to the quotas

We could look at apportionment as an optimization problem: find the seat assignment $s_1, \ldots, s_n$ with $s_1 + \cdots + s_n = h$ that minimizes the distance from the quotas, for example that minimizes

$$|s_1 - q_1| + \cdots + |s_n - q_n|$$

or that minimizes

$$(s_1 - q_1)^2 + \cdots + (s_n - q_n)^2$$

or more generally minimizes

$$|s_1 - q_1|^p + \cdots + |s_n - q_n|^p, \quad p \geqslant 1.$$

Theorem. The Method of Largest Remainder outputs the seat assignment that minimizes each of these objective functions.

Method of Largest Remainder suffers from paradoxes

The Method of Largest Remainder may seem like the most natural, even obvious, method for apportionment. But countries using it to make decisions have found that it suffers from several so-called apportionment paradoxes.

- The Alabama paradox. In 1880, the chief clerk of the U.S. Census Bureau computed apportionments for all house sizes between 275 and 350, and discovered that Alabama would get 8 seats with $h = 299$ but only 7 seats with $h = 300$.
- The population paradox. In 1900, Virginia lost a seat to Maine, even though Virginia's population was growing more rapidly.
- The new states paradox. In 1907, Oklahoma became a state and would have deserved 5 seats. So the house size was increased from 386 to 391. In the process, New York lost a seat while Maine gained a seat.

House monotonicity

An apportionment method is house monotone if for every problem $P = [h; p_1, \ldots, p_n]$ and enlarged problem $P' = [h+1; p_1, \ldots, p_n]$, writing $s_1, \ldots, s_n$ for the seat assignment of the method for problem P, and $s'_1, \ldots, s'_n$ for problem P', we have

$$s'_i \geqslant s_i \quad \text{for every state } i.$$

A failure of house monotonicity is known as the Alabama paradox.

Theorem. The Method of Largest Remainder fails house monotonicity.

Question: Do there exist any house monotone methods? How would they work?

House monotonicity is failed by Largest Remainder

Let's look at an example where the Method of Largest Remainder fails house monotonicity.

state	pop.	quota	lq	uq	remainder	seats
1	145	1.45	1	2	0.45	2
2	340	3.40	3	4	0.40	3
3	515	5.15	5	6	0.15	5
total	1000	10	9	12	1	10

state	pop.	quota	lq	uq	remainder	seats
1	145	1.595	1	2	0.595	1
2	340	3.740	3	4	0.740	4
3	515	5.665	5	6	0.665	6
total	1000	**11**	9	12	1	**11**

Population paradox

Let's look at an example where the Method of Largest Remainder exhibits what has been called the population paradox.

state	pop.	quota	lq	uq	remainder	seats
1	145	1.45	1	2	0.45	2
2	340	3.40	3	4	0.40	3
3	515	5.15	5	6	0.15	5
total	1000	10	9	12	1	10

state	pop.	quota	lq	uq	remainder	seats
1	$147^{\uparrow}$	1.55	1	2	0.55	1
2	$338^{\downarrow}$	3.56	3	4	0.56	4
3	$465^{\downarrow}$	4.89	4	5	0.89	5
total	950	10	8	11	2	10

A growing state loses a seat to a shrinking state.

Jefferson's method

U.S. constitution, Article 1, says:

> *Representatives … shall be apportioned among the several states … according to their respective numbers. … The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative.*

Thomas Jefferson devised an apportionment method based on what he understood were the principles behind this passage.

Jefferson's method

Jefferson's original method takes a fixed divisor D which is the number of people that get 1 representative, e.g. $D = 30\ 000$.

His method then assigns $s_i = \lfloor p_i/D \rfloor$ seats to each state i. Jefferson rounds the values p_i/D down, because the constitution gives an upper bound on the number of representatives per number of people.

Note that there is no reason to think that the values s_i sum to h; Jefferson did not consider the house size fixed, but the divisor fixed. [Given population growth since 1780, it would have been a bad idea to keep D at 30k throughout; there would now be 11 000 representatives instead of 435.]

Idea: to get to a fixed house size h, vary the divisor D!

Jefferson's method / D'Hondt method / Method of Greatest Divisor

This idea leads to the following rule:

Given a problem $[h; p_1, \ldots, p_n]$ find some divisor $D > 0$ such that

$$\lfloor \tfrac{p_1}{D} \rfloor + \cdots + \lfloor \tfrac{p_n}{D} \rfloor = h.$$

Then the output seat assignment is $s_i = \lfloor \frac{p_i}{D} \rfloor$.

Such a divisor D always exists (if we handle ties reasonably) and we call it the Jefferson divisor. All divisors D for which the equality holds lead to the same seat assignment, so the method is well-defined.

How to find D? Start with $D = \sum_{i=1}^{n} p_i / h$. Then the seats will sum up to less than h. While the seats sum up to less than h, decrease D. If they sum up to more than h, increase D.

Jefferson's method: Example

Suppose the target house size is $h = 10$.

		$D = 10$		$D = 8$		$D = 8.5$	
state	pop.	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$
1	15	1.50	1	1.87	1	1.76	1
2	32	3.20	3	4.00	4	3.76	3
3	53	5.30	5	6.62	6	6.24	6
total	100		9		11		10

Jefferson's method is house monotone

We can easily see that Jefferson's method is house monotone.

For a problem $P = [h; p_1, \ldots, p_n]$, suppose D is the Jefferson divisor, so $s_i = \lfloor \frac{p_i}{D} \rfloor$.

Next consider $P' = [h+1; p_1, \ldots, p_n]$. Note that

$$\lfloor \tfrac{p_1}{D} \rfloor + \cdots + \lfloor \tfrac{p_n}{D} \rfloor = h < h + 1.$$

So the seats sum up to too few seats with the divisor D. Thus we need to decrease D to make the sum larger. Thus, the Jefferson divisor D' for problem P' satisfies $D' < D$.

Thus, for every state i, $s_i' = \lfloor \frac{p_i}{D'} \rfloor \geqslant \lfloor \frac{p_i}{D} \rfloor = s_i$. This is what house monotonicity requires. $\square$

Jefferson's method satisfies lower quota

Jefferson's method satisfies lower quota.

Take a problem $P = [h; p_1, \dots, p_n]$ and write it in quota form $P' = [h; q_1, \dots, q_n]$, i.e. with populations rescaled to sum to h.

Write $D^* = \sum_{i=1}^{n} p_i/h$. Note that $q_i = D^* \cdot p_i$.

Thus, running Jefferson's method on P with divisor D^* would lead seat counts $s_i^* = \lfloor q_i \rfloor$, which would sum up to less than h:

$$\lfloor q_1 \rfloor + \cdots + \lfloor q_n \rfloor \leqslant q_1 + \cdots + q_n = h.$$

Therefore the Jefferson divisor D must be smaller than D^*.

Hence $s_i = \lfloor p_i/D \rfloor \geqslant \lfloor p_i/D^* \rfloor = \lfloor q_i \rfloor$, and thus Jefferson's method satisfies lower quota.

Jefferson's method: population monotonicity

Population monotonicity: For two problems P and P', we have

$$s_i < s_i' \text{ and } s_j > s_j' \implies p_i < p_i' \text{ or } p_j > p_j'.$$

Theorem: Jefferson's method satisfies population monotonicity.

Proof. Write D and D' for the Jefferson divisors for problems P and P'. If $s_i < s_i'$ then we must have $p_i/D < p_i'/D'$. If $s_j > s_j'$ then we must have $p_j/D > p_j'/D'$. Rearranging, we deduce

$$p_i' > \frac{D'}{D} \cdot p_i \text{ and } p_j' < \frac{D'}{D} \cdot p_j.$$

- If $\frac{D'}{D} \geqslant 1$ then $p_i' > p_i$.
- If $\frac{D'}{D} \leqslant 1$ then $p_j' < p_j$. □

Jefferson's method: an equivalent definition

For each state i with population p_i, we can write down the values of D for which i gets at least 1, at least 2, etc. seats:

$$\begin{aligned} i \text{ gets at least 1 seat when using } D &\iff \lfloor \frac{p_i}{D} \rfloor \geqslant 1 \\ &\iff \frac{p_i}{D} \geqslant 1 \\ &\iff D \leqslant p_i. \end{aligned}$$

More generally,

$$\begin{aligned} i \text{ gets at least } k \text{ seats when using } D &\iff \lfloor \frac{p_i}{D} \rfloor \geqslant k \\ &\iff D \leqslant \frac{p_i}{k}. \end{aligned}$$

Jefferson's method: an equivalent definition

We saw that

$$i \text{ gets at least } k \text{ seats when using } D \iff D \leqslant \frac{p_i}{k}.$$

Thus, if I tell you D, you can figure out how many seats i gets by counting how many of the following numbers are larger than D:

$$\frac{p_i}{1}, \frac{p_i}{2}, \frac{p_i}{3}, \frac{p_i}{4}, \frac{p_i}{5}, \ldots, \frac{p_i}{h}.$$

This suggests the equivalent "table definition" of Jefferson's method: write down the above values for each state (obtaining $n \cdot h$ numbers). Select the h largest numbers on the list. If you selected s_i of the numbers belonging to state i, then i gets s_i seats.

Jefferson's method: table definition - example

Previous example with $h = 10$:

		$D = 10$		$D = 8$		$D = 8.5$	
state	pop.	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$
1	15	1.50	1	1.87	1	1.76	1
2	32	3.20	3	4.00	4	3.76	3
3	53	5.30	5	6.62	6	6.24	6
total	100		9		11		10

Table definition:

state	pop.	$\frac{p_i}{1}$	$\frac{p_i}{2}$	$\frac{p_i}{3}$	$\frac{p_i}{4}$	$\frac{p_i}{5}$	$\frac{p_i}{6}$	$\frac{p_i}{7}$
1	15	**15**	7.5	5	3.75	3	2.5	2.14
2	32	**32**	**16**	**10.67**	8	6.4	5.33	4.57
3	53	**53**	**26.5**	**17.67**	**13.2**	**10.6**	**8.83**	7.57

Jefferson's method: house monotonicity

With the equivalent definition, house monotonicity is obvious. We just select an additional value in the table — note that the numbers in the table do not change when h changes.

state	pop.	$\frac{p_i}{1}$	$\frac{p_i}{2}$	$\frac{p_i}{3}$	$\frac{p_i}{4}$	$\frac{p_i}{5}$	$\frac{p_i}{6}$	$\frac{p_i}{7}$
1	15	**15**	7.5	5	3.75	3	2.5	2.14
2	32	**32**	**16**	**10.67**	8	6.4	5.33	4.57
3	53	**53**	**26.5**	**17.67**	**13.2**	**10.6**	**8.83**	7.57

Jefferson's method favors large states

Example 1: $(h = 10)$

state	pop.	$q_i = p_i/10$	Largest Rem.	$p_i/9.1$	Jefferson
1	18	1.8	2	1.98	1
2	82	8.2	8	9.02	9

Jefferson's method favors large states

Example 1: $(h = 10)$

state	pop.	$q_i = p_i/10$	Largest Rem.	$p_i/9.1$	Jefferson
1	18	1.8	2	1.98	1
2	82	8.2	8	9.02	9

Example 2: $(h = 10)$

state	pop.	$q_i = p_i/10$	Largest Rem.	$p_i/8$	Jefferson
1	15	1.5	2	1.88	1
2	14	1.4	1	1.75	1
3	13	1.3	1	1.62	1
4	58	5.8	6	7.25	7

Theorem: Jefferson's method fails upper quota.

Jefferson's method: another equivalent definition

Here is another equivalent definition for curiosity. It treats each person as an individual who would like to be represented by as many representatives as possible.

Let's say that an individual from state i, when it gets s_i seats, has utility

$$u(s_i) := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{s_i}.$$

Now consider the seat assignment $(s_1, \ldots, s_n)$ (summing to h) that maximizes

$$p_1 \cdot u(s_1) + p_2 \cdot u(s_2) + \cdots + p_n \cdot u(s_n).$$

This turns out to give the same apportionments as Jefferson's rule.

You'll see this objective function again later in the course, then called "Proportional Approval Voting".

Divisor methods

One can generalize the idea behind Jefferson's method (based on rounding down) to get other apportionment methods, based on other rounding functions.

A rounding function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{N}$ is a weakly increasing function (if $x \leqslant y$ then $f(x) \leqslant f(y)$) such that $f(n) = n$ for each integer $n \in \mathbb{N}$.

Jefferson's rule uses $f(x) = \lfloor x \rfloor$.

If f is a rounding function, the divisor method based on f is an apportionment method that works as follows: Given a problem $[h; p_1, \ldots, p_n]$, find a divisor D such that

$$f\left(\tfrac{p_1}{D}\right) + \cdots + f\left(\tfrac{p_n}{D}\right) = h.$$

Then return the seat assignment with $s_i = f\left(\frac{p_i}{D}\right)$ for every state i.

The 5 historic divisor methods

Every rounding function f can be specified by its dividing points $d_1, d_2, \ldots$, where d_k is the lowest value (infimum) of x such that $f(x) \geqslant k$. For example $f(x) = \lfloor x \rfloor$ has dividing points 1, 2, 3, ...

- Jefferson's method (D'Hondt): 1, 2, 3, ..., $k+1$
- Webster's method (Sainte-Laguë): 0.5, 1.5, 2.5, ..., $k+\frac{1}{2}$
- Hill's method: 0, 1.414, 2.449, 3.464, ..., $\sqrt{k(k+1)}$
- Dean's method: 0, 1.333, 2.400, 3.429, ..., $k(k+1)/(k+\frac{1}{2})$
- Adam's method: 0, 1, 2, 3, ..., k

Properties of divisor methods

- Jefferson's method is the only divisor method that satisfies lower quota.
- Adam's method is the only divisor method that satisfies upper quota.

Hence, no divisor method is a quota method.

- Every divisor method satisfies house monotonicity.
- Every divisor method satisfies population monotonicity.

The proofs are the same as for Jefferson, replacing $\lfloor \cdot \rfloor$ by $f(\cdot)$ everywhere.

- Every divisor method has an equivalent "table definition", based on a table filled with the numbers p_i/d_1, p_i/d_2, p_i/d_3.

Why these? Huntington's criteria

Apportionment aims to faithfully convert populations to seats: the two vectors should be as proportional to each other as possible.

It would be bad for an apportionment method to select an outcome where state i gets many more seats per capita than does state j:

$$\frac{s_i}{p_i} \gg \frac{s_j}{p_j}. \tag{bad}$$

Thus, a natural measure of inequality between states would be

$$\left| \frac{s_i}{p_i} - \frac{s_j}{p_j} \right|.$$

One way to implement this idea is to look for an apportionment $(s_1, \ldots, s_n)$ that is stable, where there is no way to transfer a seat from one state to another and thereby reducing inequality:

$$\text{for all } i, j: \quad \left| \frac{s_i - 1}{p_i} - \frac{s_j + 1}{p_j} \right| \geqslant \left| \frac{s_i}{p_i} - \frac{s_j}{p_j} \right|. \tag{stability}$$

Why these? Huntington's criteria

$$\text{for all } i,j: \quad \left| \frac{s_i - 1}{p_i} - \frac{s_j + 1}{p_j} \right| \geqslant \left| \frac{s_i}{p_i} - \frac{s_j}{p_j} \right|. \qquad \text{(stability)}$$

Does a stable apportionment always exist?
Rewrite the definition: for all pairs i, j with $\frac{s_i}{p_i} \geqslant \frac{s_j}{p_j}$, we must have

$$\frac{s_j + 1}{p_j} - \frac{s_i - 1}{p_i} \geqslant \frac{s_i}{p_i} - \frac{s_j}{p_j} \iff \frac{p_i}{s_i - \frac{1}{2}} \geqslant \frac{p_j}{s_j + \frac{1}{2}}.$$

The latter is the defining inequality for apportionments selected by Webster's method (e.g., it says that if in the Webster table, we select only the highest entries, we get a stable outcome). So stable apportionments exists, and the rule selecting a stable apportionment turns out to be house and population monotonic!

Question: Is this a convincing argument for using Webster?

Why these? Huntington's criteria

Each of the five historic apportionment methods are stable under a different inequality measure (written for all i, j with $\frac{s_i}{p_i} \geqslant \frac{s_j}{p_j}$).

Adams	Dean	Hill	Webster	Jefferson
$s_i - s_j \frac{p_i}{p_j}$	$\frac{p_j}{s_j} - \frac{p_i}{s_i}$	$\frac{s_i/p_i}{s_j/p_j} - 1$	$\frac{s_i}{p_i} - \frac{s_j}{p_j}$	$s_i \frac{p_j}{p_i} - s_j$

(There are 16 ways of rewriting $\frac{s_i}{p_i} > \frac{s_j}{p_j}$. 4 of the 16 do not always have stable apportionments. The rest lead to one of the 5 methods.)

Huntington: It is better to compare numbers by relative difference:

$$\frac{|y - z|}{\min(y, z)}.$$

Comparing $\frac{s_i}{p_i}$ and $\frac{s_j}{p_j}$ using relative difference in each of the 16 ways all lead to Hill's method!

Bias; ranking by favoring small states

We have seen that Jefferson's method seems to favor large states. Similar examples show that Adam's method favors small states.

In 1929, U.S. congress asked the National Academy of Sciences for an opinion. Its report favored Hill's method because it minimizes relative differences and "occupies mathematically a neutral position with respect to the emphasis on larger and smaller states".

Let's say that method M favors small states relative to method M', written $M \lhd M'$, if for all problems $[h; p_1, \ldots, p_n]$, the seat counts (s_i) and (s'_i) of these methods satisfy

$$p_i < p_j \quad \text{implies either} \quad s'_i \geq s_i \quad \text{or} \quad s'_j \leq s_j.$$

Not all pairs of methods can be compared in this way, but:

$$\text{Adams} \lhd \text{Dean} \lhd \text{Hill} \lhd \text{Webster} \lhd \text{Jefferson}.$$

Data suggests that Webster is the least biased in either direction.

Merger incentives

Another way to compare the different divisor methods comes from the case where we use apportionment to assign seats to parties in proportion to their vote counts.

Suppose a country uses D'Hondt's method (= Jefferson), as Austria does for example. We know this method favors large parties. Therefore, two parties may try to merge to get more seats.

This can work:

$$[10; 10, 8, 5, 2] \mapsto (5, 3, 2, 0) \quad \text{while} \quad [10; 10, 8, 7] \mapsto (4, 3, 3)$$

In fact, it can never hurt! D'Hondt encourages mergers:

- If $M[h; p_1, p_2, p_3, \ldots, p_n] \mapsto (s_1, s_2, s_3 \ldots, s_n)$ and $M[h; p_1 + p_2, p_3, \ldots, p_n] \mapsto (s_1', s_3' \ldots, s_n')$, then $s_1' \geqslant s_1 + s_2$.

Merger incentives

D'Hondt encourages mergers (and is the unique divisor method doing so):

- If $M[h; p_1, p_2, p_3, \ldots, p_n] \mapsto (s_1, s_2, s_3 \ldots, s_n)$ and $M[h; p_1 + p_2, p_3, \ldots, p_n] \mapsto (s_1', s_3' \ldots, s_n')$, then $s_1' \geqslant s_1 + s_2$.

Similarly, Adam's discourages mergers (and is the unique divisor method doing so):

- If $M[h; p_1, p_2, p_3, \ldots, p_n] \mapsto (s_1, s_2, s_3 \ldots, s_n)$ and $M[h; p_1 + p_2, p_3, \ldots, p_n] \mapsto (s_1', s_3' \ldots, s_n')$, then $s_1' \leqslant s_1 + s_2$.

Arguments in favor of divisor methods

- The only population monotonic rules that we have seen. (In fact, if you appropriately strengthen population monotonicity to handle ties, and add some background assumptions, then one can prove that the divisor methods are exactly the methods satisfying population monotonicity.)
- They satisfy coherence: each part of a fair solution should be fair. Suppose that a method says

  $$[h; p_1, p_2, \ldots, p_n] \mapsto (s_1, s_2, \ldots, s_n).$$

  Then on the subproblem $[s_1 + s_2; p_1, p_2]$, the rule must select (s_1, s_2). (This property also characterizes divisor methods, under background assumptions.)

Can quota methods do better?

So far we have seen only 1 quota method, the Method of Largest Remainder. It violates both house monotonicity and population monotonicity. On the other hand, divisor methods satisfy these properties but are not quota methods.

Can we find a better quota method?

Attempt: Lowndes's method: Give each state its lower quota.then assign additional seats in descending order of the quantity

$$\frac{q_i - \lfloor q_i \rfloor}{q_i}.$$

But this also fails house and population monotonicity. (Similarly for many other attempts following this scheme.)

A house monotone quota method

Balinski and Young (1975) discovered a quota method that is house monotone.

Consider the table definition of Jefferson's method. Write the table containing the entries p_i/k for each state i. Now count the h largest numbers, but skip them if selecting the number would violate upper quota.

Example: $[12; 7, 22, 71]$.

A house monotone quota method

h	critical divisors			Jefferson			standard quotas			B-Y		
	A	B	C	A	B	C	A	B	C	A	B	C
0				0	0	0				0	0	0
1	7	22	71	0	0	1	0.07	0.22	0.71	0	0	1
2	7	22	35.5	0	0	2	0.14	0.44	1.42	0	0	2
3	7	22	23.7	0	0	3	0.21	0.66	2.13	0	0	3
4	7	22	17.8	0	1	3	0.28	0.88	2.84	0	1	3
5	7	11	17.8	0	1	4	0.35	1.10	3.55	0	1	4
6	7	11	14.2	0	1	5	0.42	1.32	4.26	0	1	5
7	7	11	11.8	0	1	6	0.49	1.54	4.97	0	2	5
8	7	11	10.1	0	2	6	0.56	1.76	5.68	0	2	6
9	7	7.33	10.1	0	2	7	0.63	1.98	6.39	0	2	7
10	7	7.33	8.88	0	2	8	0.70	2.20	7.10	0	2	8
11	7	7.33	7.89	0	2	9	0.77	2.42	7.81	0	3	8
12	7	7.33	7.1	0	3	9	0.84	2.64	8.52	0	3	9
13	7	5.5	7.1	0	3	10	0.91	2.86	9.23	0	3	10
14	7	5.5	6.45	1	3	10	0.98	3.08	9.94	1	3	10
15	3.5	5.5	6.45	1	3	11	1.05	3.30	10.65	1	3	11

Balinski–Young impossibility theorem

Theorem: There exists no quota method that is population monotone.

Recall that an apportionment method is population monotone if when we consider two problems P and P', then

$$s_i < s_i' \text{ and } s_j > s_j' \implies p_i < p_i' \text{ or } p_j > p_j'.$$

Balinski–Young proved a weaker statement that also assumed order-preservation: if $p_i \geqslant p_j$ then $s_i \geqslant s_j$ (larger states get more seats). This is a mild condition and makes the proof much easier. But it was later shown that this extra assumption is not necessary.

Balinski–Young impossibility theorem: Proof

Theorem: No quota method is population monotone.

Consider house size $h = 10$.

State	pop.	q_i
1	699	6.99
2	52	0.52
3	50	0.50
4	199	1.99

By upper quota, $s_1 \leqslant 7$ and $s_4 \leqslant 2$. Thus either state 2 or 3 gets at least 1 seat. By order preservation, $s_2 \geqslant 1$.

State	pop.	q_i
1	680	8.02
2	55	0.65
3	56	0.66
4	57	0.67

By lower quota, $s'_1 \geqslant 8$. Thus, not all of states 2,3,4 can get a seat. By order preservation, $s'_2 = 0$.

We have constructed an example with $s'_1 > s_1$ and $s'_2 < s_2$ yet $p'_1 < p_1$ and $p'_2 > p_2$. $\square$

Biapportionment

Some countries face a double apportionment problem: they want to apportion parliament seats to representatives from different states in proportion to state population, and simultaneously apportion seats to parties in proportion to (nationwide) party vote count.

Not a good idea to treat each state separately: Assume we have 100 districts, and each district elects 5 candidates, so $h = 500$. Suppose there are two parties. In each district, party A gets 15 votes and party B gets 32 votes.

If we use D'Hondt's method in each district, 1 seat goes to party A and 4 seats go to party B. So party A gets 100 seats, while it deserves

$$\frac{15}{15+32} \cdot 500 \approx 160.$$

Biapportionment

Input to a biapportionment problem: a vote matrix, a desired seat count for each row, and a desired seat count for each column. (Can get desired seat counts using an apportionment method applied to the sums of each row/column.)

$$
\begin{array}{cccc|c}
v_{11} & v_{12} & \cdots & v_{1n} & r_1 \\
v_{21} & v_{22} & \cdots & v_{2n} & r_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{mn} & r_m \\
\hline
c_1 & c_2 & \cdots & c_n & h
\end{array}
$$

Biapportionment

Main idea: apply rounding function to each value in the matrix.

$$
\begin{array}{cccc|c}
f(v_{11}) & f(v_{12}) & \cdots & f(v_{1n}) & r_1 \\
f(v_{21}) & f(v_{22}) & \cdots & f(v_{2n}) & r_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f(v_{m1}) & f(v_{m2}) & \cdots & f(v_{mn}) & r_m \\
\hline
c_1 & c_2 & \cdots & c_n & h
\end{array}
$$

But this will violate the desired seat counts. Thus, we need to rescale rows and columns.

Biapportionment

Biapportionment divisor method: find divisors for each row and each column; then round.

$$
\begin{array}{cccc|c|c}
f(v_{11}/D_1E_1) & f(v_{12}/D_1E_2) & \cdots & f(v_{1n}/D_1E_n) & r_1 & D_1 \\
f(v_{21}/D_2E_1) & f(v_{22}/D_2E_2) & \cdots & f(v_{2n}/D_2E_n) & r_2 & D_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
f(v_{m1}/D_mE_1) & f(v_{m2}/D_mE_2) & \cdots & f(v_{mn}/D_mE_n) & r_m & D_m \\
\hline
c_1 & c_2 & \cdots & c_n & h & \\
\hline
E_1 & E_2 & \cdots & E_n & &
\end{array}
$$

Theorem (ignoring ties): There exist row and column divisors such that all the desired row and column totals are respected. The resulting seat assignment is unique.

Random apportionment

Given a problem, return a lottery over seat assignments.

- A lottery is ex ante proportional if each state gets q_i seats in expectation.
- A lottery satisfies ex post quota if every seat assignment with positive probability satisfies both lower and upper quota.

Question: Does such a lottery always exist?

Question: Can we define house/population monotonicity in this setting?

Allocation of indivisible items with weighted agents

- $N = \{1, \ldots, n\}$, a set of agents, each with a weight $w_i > 0$.
- $O = \{o_1, \ldots, o_m\}$, a set of indivisible objects.
- Additive valuations $u_i : O \to \mathbb{R}_{\geqslant 0}$.
- Task: find an allocation $(A_1, \ldots, A_n)$ of pairwise disjoint bundles.

An allocation A satisfies weighted EF1 if for all $i, j \in N$ with $A_j \neq \emptyset$, there exists an item $o \in A_j$ such that

$$\frac{u_i(A_i)}{w_i} \geqslant \frac{u_i(A_j \setminus \{o\})}{w_j}.$$

An allocation A satisfies weighted PROP1 if for all $i \in N$ there exists an item $o \notin A_i$ such that

$$u_i(A_i) + u_i(o) \geqslant \frac{w_i}{w_1 + \cdots + w_n} \cdot u_i(O).$$

Allocation of indivisible items with weighted agents

Using a house monotone apportionment method to obtain an assignment.

Consider the problem $[m; w_1, \dots, w_n]$. Applying a house monotone apportionment method, we obtain a sequence of agents $i_1, i_2, i_3, \dots, i_m$, with repetitions. We can now take this as a picking sequence: we go through the sequence and let each agent pick their favorite item that has not been picked yet.

- If we use Adam's method ($f(x) = \lceil x \rceil$), we obtain an allocation that satisfies weighted EF1.
- If we use Jefferson's method ($f(x) = \lfloor x \rfloor$), we obtain an allocation that satisfies weighted PROP1.

These are the unique divisor methods giving the respective guarantee.

Other topics

- Random apportionment
- Degressive proportionality (European parliament)