Optimized Democracy

Spring 2021, Lecture 9, 2021-02-24

Participatory Budgeting

Dominik Peters, Harvard University
Outline

• Aside on Single Transferable Vote
• More on approval-based committee elections: Phragmén’s rule
• Participatory Budgeting

• Repeating theme: Can get proportionality by explicitly dividing “voting power” equally among voters. (Rather than magically proportional PAV.)
Single Transferable Vote
Single Transferable Vote for Committees

• STV can also be used to elect a $k$-committee.
• Initially, each voter gets a ‘budget’ of $1$.
• It costs $\frac{n}{k}$ to elect a candidate.
• As long as there is a candidate that is ranked first by voters who together have at least $\frac{n}{k}$, elect the candidate and charge those voters $\frac{n}{k}$.
• Otherwise, eliminate the candidate whose supporters are poorest, and repeat.
• Exercise: Show STV elects $k$ candidates.
Proportionality for Solid Coalitions (PSC)

• Suppose a set $S \subseteq N$ with $|S| \geq \ell \frac{n}{k}$ has the same set $T \subseteq C$ of $|T| = t \geq \ell$ candidates they all rank top, so $T \succ C \setminus T$ for all $i \in S$ (not necessarily ranked in the same order).

• Then $|W \cap T| \geq \ell$.

• STV satisfies this! (no matter how spending is distributed)
STV satisfies PSC

- Let $S \subseteq N$ with $|S| \geq \ell \frac{n}{k}$ agree on $|T| = t \geq \ell$ candidates.
- Suppose PSC failed for $S$. Then there is a time when
  - $\ell - j$ candidates from $T$ have been elected
  - $j$ further candidates from $T$ need to be elected for PSC
  - all but $j$ candidates from $T$ have been elected or eliminated.
- Group $S$ has only paid at most $\$(\ell - j) \frac{n}{k}$ thus far, so has $\$j \frac{n}{k}$ left over. So at least one of the $j$ candidates has $\$\frac{n}{k}$ support, and this will remain true until all $j$ candidates have been elected.
Hare vs Droop Quota

• The value \(\frac{n}{k}\) is known as the **Hare quota**.
  • Intuition: electorate is split into equal-sized groups, each of which is assigned one seat.

• But we can also use \(\frac{n}{k+1} + \varepsilon\), the **Droop quota**.
  • This works because there are at most \(k\) disjoint subsets of \(N\) of size \(\frac{n}{k+1} + \varepsilon\).
  • Guarantees representation to smaller groups.
  • For \(k = 1\), this says majority needs to be followed.

• *Everything we’ve said works for Droop quota if we are more careful in the proofs.*
  • PAV satisfies Droop EJR, Droop-STV satisfies Droop PSC
Open Problem

Does there exist a ranking-based committee rule that is monotonic and satisfies PSC?
Recap: Approval-based Committee Elections

• Proportional Approval Voting maximizes
  \[ \sum_{i} \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{|W \cap A_i|}. \]

• PAV satisfies Extended Justified Representation:
  If \( S \subseteq N \) with \( |S| \geq \ell \frac{n}{k} \) agrees on \( \ell \) candidates
  \( T \subseteq \bigcap_{i \in S} A_i \), then \( |W \cap A_i| \geq \ell \) for some \( i \in S \).

• PAV is NP-complete to compute.

• Sequential PAV fails EJR even for \( \ell = 1 \).

• Question: Can we get something proportional in polynomial time?
Is PAV always right?

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$k = 12$

EJR not strong enough to capture this!
Phragmén’s Rule

• Proposed in 1894. Thiele proposed PAV in 1895. Phragmén criticized it in 1899, for a reason similar to

• Phragmén’s proposal:
  • Each voter starts with a bank account with $0.
  • Fill bank accounts at the same rate, until the approvers of some unelected candidate together hold $\frac{n}{k}$.
  • Elect the candidate and reset approvers’ accounts to $0$.
  • Stop after $k$ candidates are elected.
Phragmén’s Rule: Example

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$k = 12$

Sveriges Riksbank
Phragmén’s Rule: Example

$k = 12$

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Sveriges Riksbank
Phragmén’s Rule: Example

$k = 12$

Sveriges Riksbank

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Phragmén’s Rule: Example

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Phragmén’s Rule: Example

\[ k = 12 \]

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Sveriges Riksbank
Phragmén’s Rule: Example

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Phragmén’s Rule: Example

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$k = 12$

Sveriges Riksbank

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Phragmén’s Rule: Example

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$Sveriges Riksbank$

$k = 12$

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Phragmén’s Rule: Example

\[ k = 12 \]

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Phragmén’s Rule: Proportionality

• Phragmén’s rule violates EJR (largish example with 24 voters, 14 candidates, $k = 12$).

• But it satisfies a weaker version (“PJR”):
  If $S \subseteq N$ with $|S| \geq \ell \frac{n}{k}$ agrees on $\ell$ candidates $T \subseteq \cap_{i \in S} A_i$, then $|W \cap \bigcup_{i \in S} A_i| \geq \ell$.

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$v_1$, $v_2$, $v_3$, $v_4$, $v_5$
Phragmén’s Rule: PJR

• If $S \subseteq N$ with $|S| \geq \ell \frac{n}{k}$ agrees on $\ell$ candidates $T \subseteq \bigcap_{i \in S} A_i$, then $|W \cap \bigcup_{i \in S} A_i| \geq \ell$.

• Proof: By the time Phragmén terminates, each voter has received at least $\$1$.

• If it terminates exactly at the $\$1$-point, then all money was spent. So $S$ spent $\$\ell \frac{n}{k}$, and so they bought $\ell$ candidates from $\bigcup_{i \in S} A_i$.

• If it terminates strictly later, consider $\$1$-point. If then $|W \cap \bigcup_{i \in S} A_i| \leq \ell - 1$, then $S$ now holds at least $\$\frac{n}{k}$, so can purchase a candidate from $T$. 
Proportional Rankings

• Note: you don’t have to stop Phragmén after it has elected $k$ candidates (same for SeqPAV)
• This way, we get a proportional ranking.
• In particular, every prefix satisfies PJR. (Or think of party-list profiles.)
• Applications:
  • Ranking comments by upvotes
  • Displaying proposal variants in LiquidFeedback
• Open Problem: Do there exist EJR rankings?
**BCYF Hyde Park Dance Studio Renovation**
A renovated dance studio at the Hyde Park Community Center for children of all ages.

**Estimated Cost:** $286,000

**Location:** BCYF Hyde Park Community Center, Hyde Park

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**Bike Lane Installation**
After a study, bike lanes will begin to be installed around Charlestown Navy Yard, Bunker Hill housing, and Charlestown High.

**Estimated Cost:** $200,000

**Location:** Charlestown

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**Wicked Free Wifi 2.0**
Wicked free Wi-Fi 2.0 provides Wi-Fi at locations with young people.

**Estimated Cost:** $119,000

**Location:** Various High Schools and Community Centers, Dorchester, Roxbury, East Boston, Charlestown

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Click image for slide show
## Education

### Bathroom Renovations at M.S./H.S. 223
Renovation of girls' and boys' bathrooms including stalls, lighting, painting, and having walls re-glazed.

**Estimated Cost:** $150,000  
**Location:** 360 E.145th St. (Bronx - Mott Haven)

- [✓ Selected](#)  
- [Remove](#)

### Technology Upgrades
Technology upgrades for Park East High School and Central Park East High School.

**Estimated Cost:** $312,000  
**Location:** 230 E.105th; 1573 Madison Ave. (El Barrio/East Harlem)

- [Select](#)

### Air Conditioning at Bronx Schools
Installation of 1 air conditioning system at 345 Brooke Avenue for schools X343, X224 & X334. Installation of 1 air conditioning system at PS 161x.

**Estimated Cost:** $500,000  
**Location:** 628 Tinton Ave, 345 Brook Ave. (Bronx - Mott Haven & Longwood)

- [✓ Selected](#)  
- [Remove](#)

### Air Conditioning: P.S.179, P.S.369, P352
Installation of two HVAC units at P.S. 179X, P.S. 369X and P.S. 352X.

**Estimated Cost:** $500,000  
**Location:** 468 E. 140 St. (Bronx - Mott Haven)
### 4e arrondissement

Greedy: total utility 3500. Funds 5 projects, avg cost 293 000  tree  tree  tree  tree  tree  
Optimal: total utility 6878. Funds 14 projects, avg cost 98 928  tree  tree  tree  tree  tree  tree  tree  tree  tree  tree  tree  tree  tree  tree

<table>
<thead>
<tr>
<th>Project Name</th>
<th>QPOP</th>
<th>Cost €</th>
<th>Votes</th>
<th>v / k€</th>
<th>Greedy</th>
<th>Optimal</th>
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</thead>
<tbody>
<tr>
<td>Un mur végétalisé au croisement des rues Blancs Manteaux et Archiv</td>
<td>30 000</td>
<td>788</td>
<td>26</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
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<tr>
<td>Un café solidaire dans le quartier de la tour Saint-Jacques</td>
<td>15 000</td>
<td>706</td>
<td>47</td>
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<td>✔</td>
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<td>Une salle d'arts plastiques pour l'école Saint-Merri Renard</td>
<td>300 000</td>
<td>702</td>
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<td>✔</td>
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<td>Rénovation énergétique exemplaire d'une école du 4e</td>
<td>1 000 000</td>
<td>655</td>
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<td>–</td>
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<td>Végétalisation de la rue de l'Arsenal</td>
<td>120 000</td>
<td>649</td>
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<td>✔</td>
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<td>Un collège Charlemagne accessible aux personnes à mobilité réduit</td>
<td>200 000</td>
<td>630</td>
<td>3</td>
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<td>Faire du hall d'accueil de la piscine Saint-Merri un lieu de convivia</td>
<td>20 000</td>
<td>528</td>
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<tr>
<td>Des agrès sportifs place des Vosges</td>
<td>15 000</td>
<td>491</td>
<td>33</td>
<td>–</td>
<td>✔</td>
<td>✔</td>
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<td>Mise en valeur des pierres de la prison de la Bastille</td>
<td>20 000</td>
<td>473</td>
<td>24</td>
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<td>✔</td>
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<td>Un fauteuil roulant électrique pour rompre l'isolement</td>
<td>5 000</td>
<td>453</td>
<td>91</td>
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<td>Création d'un auvent sur une aire de jeux d'un square du 4e</td>
<td>150 000</td>
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<td>L'Ascenseur, un tiers-lieu pour l'égalité des chances ouvert sur le</td>
<td>350 000</td>
<td>315</td>
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<td>Baliser le passage du chemin de Compostelle dans le 4e</td>
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<td>Des arceaux vélos rue de la Reynie</td>
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<tr>
<td>Faciliter l'accès au cellier d'Ourscamp</td>
<td>120 000</td>
<td>228</td>
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Total € 1 465 000 1 385 000
Participatory Budgeting: Model

- A set $C$ of projects, each with a cost
- A budget limit $B$
- Outcome: set $W \subseteq C$ with $\sum_{c \in W} \text{cost}(c) \leq B$.
- A set $N$ of $n$ voters
- Each voter $i \in N$ approves a subset $A_i \subseteq C$.
- Mostly, we say that $i$’s utility is $u_i(W) = |A_i \cap W|$ (this is a dichotomous preference assumption).
- **Unit cost assumption**: $\text{cost}(c) = 1$ for all $c$. 
Three interpretations of “AV”

• Optimize $\sum_{i \in N} u_i(W) = \sum_{c \in W} \text{approval-score}(c)$.

• Greedy: add projects in order of approval score, skipping unaffordable projects.

• Bang-per-buck greedy: add projects in order of approval score divided by cost.
Experiments

Budget = $1000. Cheap = $10. Expensive = $10, $30, $90, $190.
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<th>optimum</th>
<th>greedy</th>
<th>bang-per-buck</th>
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Circleville

- Northside pop. 120k
- Eastside pop. 110k
- Westside pop. 90k
- Southside pop. 80k

$400,000 pop 400,000

Example River

$500,000 $300,000

$150,000 $100,000

$300,000 $300,000

$600,000 $600,000

$1,000,000 $1,000,000
same budget
same population
same district structure
same utilities
same projects
same feasible sets

$90,000
pop 90,000

Onetown

Leftside
pop. 60k

$20k
$20k
$20k

Rightside
pop. 30k

$45k

Twotown

Leftside
pop. 60k

$30k
$30k
$30k

Rightside
pop. 30k

$30k

not proportional!

Rightside deserves $30k

not proportional!

Leftside deserves $60k

{L_1, L_2, L_3} \rightarrow \text{PAV-score 110,000}

{L_1, L_2, R} \rightarrow \text{PAV-score 120,000}

not proportional!

{L_1, L_2, L_3} \rightarrow \text{PAV-score 110,000}

{L_1, L_2, R} \rightarrow \text{PAV-score 120,000}
Phragmén for PB

• Phragmén’s rule can easily be adapted:
  • Fill bank accounts
  • If the approvers of a project have enough money to finance its cost, implement the project
  • Stop when next project doesn’t fit into the budget.

• Picks correct outcome in Onetown and Twotown.

- Onetown
  - Leftside pop. 60k
    - $L_1$ $20k$
    - $L_2$ $20k$
    - $L_3$ $20k$
  - Rightside pop. 30k
    - $R$ $45k$

- Twotown
  - Leftside pop. 60k
    - $L_1$ $30k$
    - $L_2$ $30k$
    - $L_3$ $30k$
  - Rightside pop. 30k
    - $R$ $30k$

• Satisfies “PJR”: If $\frac{|S|}{n} \geq \alpha$ and $\text{cost}(\cap_{i \in S} A_i) \geq \alpha \cdot B$, then $\text{cost}((W \cap \cup_{i \in S} A_i) \cup \{c\}) \geq \alpha \cdot B$ for some $c \in \cap_{i \in S} A_i$. 
Rule X for PB

• Split the city budget evenly among residents.
• Put each resident’s share $\frac{B}{n}$ in a bank account.
• Repeatedly, until the budget runs out:
  • identify a project whose supporters have enough money left to afford it
  • charge the cost to supporters
Rule X for PB

- Split the city budget evenly among residents.
- Put each resident’s share $\frac{B}{n}$ in a bank account.
- Repeatedly, until the budget runs out:
  - always divide the cost of a project among supporters as evenly as possible
  - find an affordable project with the lowest max payment.
Rule X for PB

• Split the city budget evenly among residents.
• Put each resident’s share \( \frac{B}{n} \) in a bank account.
• Repeatedly, until the budget runs out:
  • always divide the cost of a project among supporters as evenly as possible
  • find an affordable project with the lowest max payment.

![Graph showing two projects with cost $16 each, split among voters.](image)
Rule X for PB

• Split the city budget evenly among residents.
• Put each resident’s share \( \frac{B}{n} \) in a bank account.
• Repeatedly, until the budget runs out:
  • always divide the cost of a project among supporters as evenly as possible
  • find an affordable project with the lowest max payment.

• Rule X satisfies EJR!
• Let \( \frac{|S|}{n} \geq \alpha \), and take \( T \subseteq \bigcap_{i \in S} A_i \) with \( \text{cost}(T) \leq \alpha \cdot B \).
  • Then \( u_i(W) \geq u_i(T) \) for some \( i \in N \) (i.e. \( |W \cap A_i| \geq |T| \)).
Additive Valuations

Motivating Example:
2019, Paris, 16th arrondissement
€560k: refurbish sports facility — 775 votes
€3k: materials for classroom project — 670 votes
— 1.15x as popular, 186x the cost!

• Utility of outcome: \( u_i(W) = \sum_{c \in W} v_i(c) \).
• Phragmén: no obvious way of extending to additive utilities.
• Rule X: can extend using following idea: a voter’s payment for a candidate should be proportional to the voter’s utility for the candidate.
• Core may be empty!
Core for Additive Valuations

• A group $S \subseteq N$ with $\frac{|S|}{n} \geq \alpha$ blocks $W$ if there is $T \subseteq C$ with $|T| \leq \alpha \cdot B$ such that $u_i(T) > u_i(W)$ for all $i \in S$.

<table>
<thead>
<tr>
<th></th>
<th>Voter 1</th>
<th>Voter 2</th>
<th>Voter 3</th>
<th>cost</th>
</tr>
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<tbody>
<tr>
<td>$u_i(a)$</td>
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<td>1</td>
<td>0</td>
<td>$2</td>
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<tr>
<td>$u_i(b)$</td>
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<td>2</td>
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<td>$2</td>
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<tr>
<td>$u_i(c)$</td>
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<td>2</td>
<td>$2</td>
</tr>
</tbody>
</table>

Budget $B$ $\$3$

• An approximation exists if we put $|T| \leq \frac{\alpha}{32} \cdot B$. The factor of 32 might be improvable to 2, but not further.

• Existence open for approval utilities.
Bibliography


