Precise Complexity of the Core in Dichotomous and Additive Hedonic Games

Dominik Peters

Department of Computer Science
University of Oxford

ADT – Luxembourg – 25 October 2017

Hedonic Game

- A model of **Coalition Formation**
- **Input:** Finite set N of agents; for each agent $i \in N$ a preference relation $\succsim_i$ over $\mathcal{N}_i = \{S \subseteq N : i \in S\}$.
- **Output:** A *partition* π of the agent set into disjoint *coalitions*.
- $\pi(i)$ denotes the coalition that i is in.
- **Aim:** Find a partition that is *stable*.

Stability Concepts: Nash-stability

Definition

A partition π is *Nash-stable* if there is no agent i such that

$$\pi(j) \cup \{i\} \succ_i \pi(i);$$

thus no agent wants to change coalitions.

- Variant: Individual stability only allows deviating if the new coalition welcomes i.

Stability Concepts: Core and Strict Core

Definition

A partition π is *core-stable* if there is no non-empty *blocking* coalition $S \subseteq N$ such that

$$S \succ_i \pi(i) \text{ for all } i \in S.$$

Stability Concepts: Core and Strict Core

Definition

A partition π is *core-stable* if there is no non-empty *blocking* coalition $S \subseteq N$ such that

$$S \succ_i \pi(i) \text{ for all } i \in S.$$

Definition

A partition π is *strict-core-stable* if there is no non-empty blocking coalition $S \subseteq N$ such that

$$S \succsim_i \pi(i) \text{ for all } i \in S, \text{ and}$$
$$S \succ_j \pi(j) \text{ for some } j \in S.$$

Computational Problem

- **Given hedonic game, find a stable partition.**
- In some games, there are no stable partitions, so we can consider the decision problem:
- **Given hedonic game, does there exist a stable partition?**
- How can we encode a hedonic game in the input?
  Naive encoding has exponential size! 2^{n-1} coalitions

Additive Hedonic Games

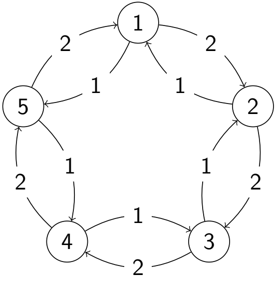
A hedonic game is **additive** if there are functions

$$v_i(j) : N \to \mathbb{R}$$

such that for coalitions $i \in S, T \subseteq N$, we have

$$S \succcurlyeq_i T \iff \sum_{j \in S} v_i(j) \geqslant \sum_{j \in T} v_i(j).$$

Can then encode input by giving the n^2 numbers $(v_i(j))_{i,j \in N}$.



Complexity Results for Additive Hedonic Games

Sung & Dimitrov (2010)	Nash		$(-\infty, +\infty)$	NP-c.
Sung & Dimitrov (2010)	(strict-)core		$(-\infty, 18]$	NP-h.
Aziz et al. (2011)	(strict-)core	symm.	$[-33, 11]$	NP-h.

Complexity Results for Additive Hedonic Games

Sung & Dimitrov (2010)	Nash		$(-\infty, +\infty)$	NP-c.
Sung & Dimitrov (2010)	(strict-)core		$(-\infty, 18]$	NP-h.
Aziz et al. (2011)	(strict-)core	symm.	$[-33, 11]$	NP-h.
Woeginger (2013)	core		$(-\infty, +\infty)$	Σ_2^p-c.

Complexity Results for Additive Hedonic Games

Sung & Dimitrov (2010)	Nash		$(-\infty, +\infty)$	NP-c.
Sung & Dimitrov (2010)	(strict-)core		$(-\infty, 18]$	NP-h.
Aziz et al. (2011)	(strict-)core	symm.	$[-33, 11]$	NP-h.
Woeginger (2013)	core		$(-\infty, +\infty)$	Σ_2^p-c.
Rey et al. (2014)	strict-core	symm.	$\{-\infty, 1\}$	DP-h.

Complexity Results for Additive Hedonic Games

Sung & Dimitrov (2010)	Nash		$(-\infty, +\infty)$	NP-c.
Sung & Dimitrov (2010)	(strict-)core		$(-\infty, 18]$	NP-h.
Aziz et al. (2011)	(strict-)core	symm.	$[-33, 11]$	NP-h.
Woeginger (2013)	core		$(-\infty, +\infty)$	Σ_2^p-c.
Rey et al. (2014)	strict-core	symm.	$\{-\infty, 1\}$	DP-h.
P. (2015/17; this paper)	(strict-)core	symm., sparse	$[-100, 30]$	Σ_2^p-c.
Ota et al. (2017)	(strict-)core		$\{-\infty, 0, 1\}$	Σ_2^p-c.

Complexity Results for Additive Hedonic Games

Sung & Dimitrov (2010)	Nash		$(-\infty, +\infty)$	NP-c.
Sung & Dimitrov (2010)	(strict-)core		$(-\infty, 18]$	NP-h.
Aziz et al. (2011)	(strict-)core	symm.	$[-33, 11]$	NP-h.
Woeginger (2013)	core		$(-\infty, +\infty)$	Σ_2^p-c.
Rey et al. (2014)	strict-core	symm.	$\{-\infty, 1\}$	DP-h.
P. (2015/17; this paper)	(strict-)core	symm., sparse	$[-100, 30]$	Σ_2^p-c.
Ota et al. (2017)	(strict-)core		$\{-\infty, 0, 1\}$	Σ_2^p-c.
open	(strict-)core	symm.	$\{-\infty, 0, 1\}$	Σ_2^p-c.?
open	strict-core	symm.	$\{-\infty, 1\}$	Σ_2^p-c.?
open (even for NP-h.)	(strict-)core	symm.	$\{-1, 0, 1\}$	Σ_2^p-c.?

Boolean Hedonic Games

- Consider *dichotomous preferences*: A coalition is either approved or not:
  $\{$ approved coalitions $\} \succ_i \{$ disapproved coalitions $\}$.

Boolean Hedonic Games

- Consider *dichotomous preferences*: A coalition is either approved or not:
  $\{$ approved coalitions $\} \succ_i \{$ disapproved coalitions $\}$.
- Concise representation of preferences via a *goal* given in propositional logic:
  Boolean Hedonic Game (Aziz et al. KR 2016)
- Example: goal for agent i is $\neg j \wedge (k \vee \ell)$
  i approves a coalition iff
  it does not contain j but contains either k or ℓ.

Boolean Hedonic Games

- Consider *dichotomous preferences*: A coalition is either approved or not:
  $\{$ approved coalitions $\} \succ_i \{$ disapproved coalitions $\}$.
- Concise representation of preferences via a *goal* given in propositional logic:
  Boolean Hedonic Game (Aziz et al. KR 2016)
- Example: goal for agent i is $\neg j \wedge (k \vee \ell)$
  i approves a coalition iff
  it does not contain j but contains either k or ℓ.
- Such games always admit a core-stable partition (though it is hard to find; Peters AAAI 2016).

Boolean Hedonic Games

- Consider *dichotomous preferences*: A coalition is either approved or not:
  $\{$ approved coalitions $\} \succ_i \{$ disapproved coalitions $\}$.
- Concise representation of preferences via a *goal* given in propositional logic:
  Boolean Hedonic Game (Aziz et al. KR 2016)
- Example: goal for agent i is $\neg j \wedge (k \vee \ell)$
  i approves a coalition iff
  it does not contain j but contains either k or ℓ.
- Such games always admit a core-stable partition (though it is hard to find; Peters AAAI 2016).
- **Theorem:** It is Σ_2^p-complete to decide the existence of a strict-core-stable partition.

RESTRICTED TRUE ∃∀-3DNF

Instance: A quantified Boolean formula of form

$$\exists x_1, \ldots, x_m \, \forall y_1, \ldots, y_n \;\; \phi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

where ϕ is in disjunctive normal form

Question: Is the formula true?

$\rightsquigarrow$ Introduce agents for each variable and each clause.

RESTRICTED TRUE ∃∀-3DNF

Instance: A quantified Boolean formula of form

$$\exists x_1, \ldots, x_m \, \forall y_1, \ldots, y_n \;\; \phi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

where ϕ is in disjunctive normal form with

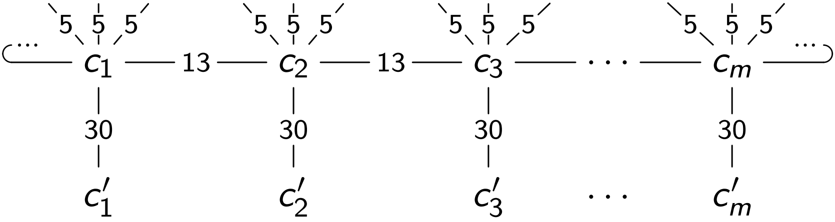
- each disjunct containing 2 or 3 literals,
- each x-variable occurring exactly once positive and once negative
- each y-variable occurring exactly three times, and at least once positively and at least once negatively.

Question: Is the formula true?

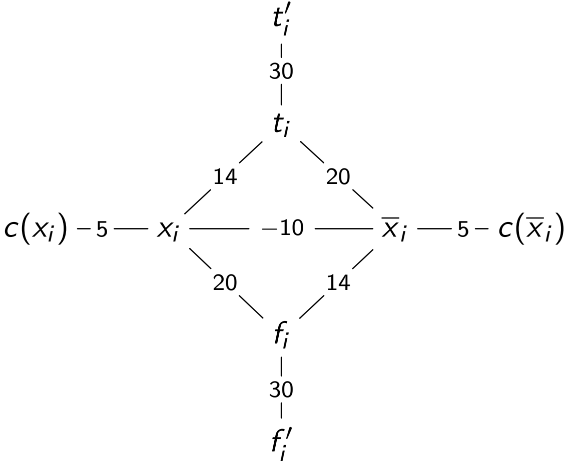
$\rightsquigarrow$ Introduce agents for each variable and each clause.

Reduction for Additive Games

Idea: Simulate the previous reduction using additive valuations.



Reduction: Variable Gadget



Sparseness of the Reduction

- The reduction produced is *sparse*: Agents have non-zero valuations for at most 10 other agents.
- All the other reductions mentioned in the table are dense (many $-\infty$ edges)
- In the language of *graphical hedonic games* (Peters AAAI 2016), this means that hardness for the core holds even for sparse graphs.
- Peters (2016): core is easy for sparse graphs of bounded treewidth $\rightsquigarrow$ we cannot remove the bounded treewidth assumption
- Open: Is the core easy also for dense graphs of bounded treewidth?

Conclusions and Future Work

- Core-stability is computationally much harder than Nash-stability!
- Root cause of Σ_2^p-hardness: large deviating coalitions
- Can define a "k-core" that only avoids deviations by coalitions of size $\leqslant k \rightsquigarrow$ problems in NP
- 2-core can be easy (stable roommates without ties), but can already be hard (with ties); 3-core will usually be hard (follows from, e.g., Peters and Elkind IJCAI 2015)
- More possible work on this.
- Our reduction: hard even for sparse graphs. What about hardness for few agent types? few allowed valuations? planar graphs? bipartite graphs?
- Are other solution concepts (e.g. Nash) also hard for sparse instances?

Precise Complexity of the Core in Dichotomous and Additive Hedonic Games

Dominik Peters

Department of Computer Science
University of Oxford

ADT – Luxembourg – 25 October 2017