Proportional Aggregation of Preferences for Sequential Decision Making

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Abstract

We study the problem of fair sequential decision making given voter preferences. In each round, a decision rule must choose a decision from a set of alternatives where each voter reports which of these alternatives they approve. Instead of going with the most popular choice in each round, we aim for proportional representation, using axioms inspired by the multi-winner voting literature. The axioms require that every group of $\alpha\%$ of the voters, if it agrees in every round (i.e., approves a common alternative), then those voters must approve at least $\alpha\%$ of the decisions. A stronger version of the axioms requires that every group of $\alpha\%$ of the voters that agrees in a β fraction of rounds must approve $\beta \cdot \alpha\%$ of the decisions. We show that three attractive voting rules satisfy axioms of this style. One of them (Sequential Phragmén) makes its decisions online, and the other two satisfy strengthened versions of the axioms but make decisions semi-online (Method of Equal Shares) or fully offline (Proportional Approval Voting). We present empirical results for these rules based on synthetic data and U.S. political elections. We also run experiments using the moral machine dataset about ethical dilemmas. We train preference models on user responses from different countries and let the models cast votes. We find that aggregating these votes using our rules leads to a more equal utility distribution across demographics than making decisions using a single global preference model.

1 Introduction

We consider the problem of making a sequence of independent decisions via voting. In each round, we can choose one alternative from a set of several alternatives, based on voters who tell us which alternatives they support (or *approve*). The set of voters stays the same across rounds, though the set of alternatives may change. The popular way of making such decisions is to take the alternative with the most supporters in each round. A problem with this method is that non-majority groups of voters may have very little influence on the outcomes. For example, if there is a fixed group of 51% of the voters who all report the same opinion in every round, then 100% of the decisions will be taken in accordance with the wishes of that group, with the other 49% of voters ignored or at most acting as tie-breakers. In many contexts, this is undesirable due to fairness concerns.

Following recent work that studies this model under the name "perpetual voting" (Lackner 2020), we define formal properties of voting rules that capture the intuition that a group of $\alpha\%$ of the voters should be able to control the decisions in $\alpha\%$ of the rounds. Inspired from work in multi-winner approval voting (Lackner and Skowron 2023), we then define a number of voting rules that satisfy these properties. We believe that these rules have promising applications in a variety of domains. Here are some examples:

- *Hiring decisions*: Consider a department that hires new faculty each year, with existing faculty voting over the applicants. The department wishes to hire people representing its spectrum of research interests. For example, if 20% of the department works on one topic and votes for candidate on that topic, then at least 1 such candidate should be hired every 5 years.
- *Virtual democracy*: In cases where a group of people need to make an extremely large number of decisions, we may wish to automate this process. An approach known as virtual democracy does this by initially learning voters' preferences over a space of potential alternatives (specified by feature vectors) for example based on pairwise comparisons. Then, each decision is made by letting the models vote on the decision maker's behalf by predicting their preferences. This approach has led to proof-of-concept systems that automate moral decisions faced by autonomous vehicles (Noothigattu et al. 2018), kidney exchanges (Freedman et al. 2020), collective decision making directly from natural language preferences (Mohsin et al. 2021) and allocation of food donations (Lee et al. 2019). The approach behind these systems has recently been criticized as overweighting the opinions of majorities (Feffer, Heidari, and Lipton 2023). We show preliminary evidence in our experiments that using proportional voting methods could avoid this issue.
- *Policy decisions of coalition governments*: In many countries, the government is formed by coalitions of several parties with different strengths (for example, in Germany 2021–25 it consists of 3 parties who received 26%, 15%, and 12% of the vote). A coalition needs to agree on a program, consisting of decisions on many issues. Our methods could help to design a program where each party's preferences gets a fair representation in the outcome.

Voting Rule	PJR	Strong PJR	EJR	Strong EJR
Sequential Phragmén (Online)	✓	✓	✗+	✗+
Method of Equal Shares (Semi-online)	✓	✗	✓	✗
PAV (Offline)	✓	✓	✓	✓

Table 1: Our results. For each setting (online, semi-online (where the rule only knows the total number of rounds in advance), offline (where the rule knows all the preferences in advance)), we list the best known rule with respect to proportionality axioms. + = knowing an online rule satisfying (Strong) EJR would resolve a major open problem in the theory of multi-winner voting.

Our Results We build on the work of Bulteau et al. (2021) who defined axioms that guarantee group representation, which we call PJR (Proportional Justified Representation) and Strong PJR. These axioms do not require the groups to be pre-defined. Instead, the guarantee applies to *all* subsets of voters who support a common alternative in every round. "Strong" axioms also apply to groups that only support a common alternative in some of the rounds. Bulteau et al. (2021) conjectured that no polynomial-time algorithm can find a PJR outcome. We show that the picture is more positive: an attractive polynomial-time voting rule building on ideas of Phragmén (1894) satisfies Strong PJR (Section 4.1). This rule is fully online, making decisions round by round.

We then define axioms called EJR (Extended Justified Representation), which provides better group representation guarantees than PJR, and analogously Strong EJR. We show that the existence of a fully online rule that satisfies EJR would positively resolve a major open problem in multi-winner approval voting (Lackner and Skowron 2023). Still, we can define a simple voting rule (a variant of the "Method of Equal Shares") that satisfies EJR and that is "semi-online": it needs to know how many rounds there will be in total, but it can be given preferences in an online fashion (Section 4.2). However, it does not satisfy Strong PJR or Strong EJR. This suggests the question whether we can satisfy Strong EJR in the offline setting, where all preferences are available in advance. In Section 4.3 we show that Proportional Approval Voting (PAV), an offline rule that maximizes a carefully chosen objective function proposed by Thiele (1895), does satisfy all our axioms, including Strong EJR. While the PAV optimization problem is NP-hard, we prove these results also hold for a polynomial-time local search variant of PAV. Finally, in Section 5 we show that further strengthenings of the Strong PJR/EJR axioms are not satisfiable, and that on inputs where a solution exists, online rules still must fail them.

We close with several experimental results on synthetic (Section 6.1) and real datasets (Section 6.2) comparing our rules to existing rules from the literature (which do not satisfy our axioms). In Section 6.3, we study the *virtual democracy* application based on the *moral machine* dataset (Awad et al. 2018). We learn separate preference models that predict the respondents judgement for each country and use each country's model as a 'voter'. We then sample new decision situations and query the models for their top choices among the alternatives in each situation, aggregating the responses using our voting rules to produce a decision sequence. We compare the decisions produced with an alternative approach where the top choices of a single universal model trained on equal number of responses from all countries are used. We find that when voters do not already have very similar preferences on the issues, the aggregation approach leads to a much fairer outcomes than the decisions made by the combined model.

Full Version In this conference paper, we have omitted some proofs and additional details due to space constraints. A full version is available (Chandak, Goel, and Peters 2023).

2 Related Work

Perpetual Voting Our work is closely related to perpetual voting (Lackner 2020) which studies online rules, while we also consider offline rules. Lackner (2020) focused on *individual* fairness properties, for example requiring that each voter approves at least 1 decision in every time interval of some bounded length. We are interested in guarantees for *groups* of voters who agree with each other, such that larger groups receive stronger guarantees. Such guarantees have been considered by Lackner and Maly (2023) and Bulteau et al. (2021); we compare them to ours in Section 3.

Public Decision Making Conitzer, Freeman, and Shah (2017) study "public decision making" which is an offline model of several decisions, where a voter's total utility is the sum of the utilities obtained in each round. We study the special case where the utility values in each round are restricted to 0 and 1. Conitzer, Freeman, and Shah (2017) focus on fairness notions for individuals (not groups) derived from fair division. Freeman, Kahng, and Pennock (2020), Skowron and Górecki (2022), and Brill et al. (2023b) consider the special case where there are exactly two alternatives ("yes/no") in each round and utilities are 0/1, which is a special case of our setting where fairness properties are easier to obtain.

Multi-winner Approval Voting For the task of selecting a committee of k out of m candidates given approval votes, proportionality has been intensely studied (see the book by Lackner and Skowron 2023). It can be considered a special case of our offline setting where voter preferences remain the same across all rounds but a different alternative (candidate) needs to be elected in each round. Many rules and axioms we consider are analogues of proposals in multi-winner voting.

Combinatorial Voting A classic literature on voting in combinatorial domains (Lang and Xia 2016) studies the problem of making decisions on several issues. The main focus is on the representation of complex preferences, and the computational problem of finding good outcomes given such preferences, often in the sense of maximizing utilitarian or egalitarian welfare (Amanatidis et al. 2015). Some of this work considers (conditional) approval preferences (Barrot and Lang 2016; Barrot, Lang, and Yokoo 2017).

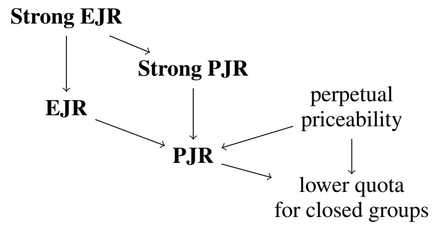

Figure 1: Implications between axioms.

3 Problem Formulation

3.1 Model

Let $N = \{1, 2, \dots, n\}$ be the set of *voters*, with $|N| = n$. There is a set $R = \{1, 2, \dots, T\}$ of T *rounds*, where T is the *time horizon*. In each round $j \in R$, we are given a set of *alternatives* C_j. Each voter $i \in N$ *approves* some (usually non-empty) subset A^i_j of C_j. Thus, we have a sequence of T decision rounds together with a sequence $C = (C_j)_{j\in R}$ of alternative sets and a collection $A = (A^i_j)_{i\in N, j\in R}$ of approval sets. We let (N, R, C, A) denote a *decision instance*.

A *decision sequence* $D = (d_1, \dots, d_T) \in C_1 \times \cdots \times C_T$ specifies a single *decision* $d_j \in C_j$ for each round $j \in R$. For a voter $i \in N$, we write $U^i_D = |\{j \in R : d_j \in A^i_j\}|$ for the number of decisions in D that i approves; we treat U^i_D as i's *utility*. A *decision rule* f takes as input a decision instance and returns a decisions sequence $D = f(N, R, C, A)$. We call a decision rule *semi-online* if its decision in round $j \in R$ only depends on the information up to round j, i.e., on N, R, $C_1, \dots, C_j$ and $A^i_1, \dots, A^i_j$. A rule is *online* if in addition, the decisions are independent of the time horizon T.

3.2 Axioms

We now define several properties (or *axioms*) of decision sequences that formalize the idea of proportional representation. These come in different strengths and Figure 1 shows implication relationships between them. We will focus on the four strong properties of (Strong) EJR and (Strong) PJR, shown in bold. For completeness, the figure also mentions two properties introduced by Lackner and Maly (2023) (perpetual priceability and lower quota compliance). We discuss these properties, as well as Pareto-efficiency, in the full version.

Our first two axioms were introduced by Bulteau et al. (2021).[1] A group $S \subseteq N$ of voters *agrees* in round $j \in R$ if there is an alternative $c \in C_j$ that all voters in S approve, so $\bigcap_{i\in S} A^i_j \neq \emptyset$. PJR (Proportional Justified Representation, first defined for multi-winner voting by Sánchez-Fernández et al. 2017) requires that a group of an α fraction of the voters, if it agrees in every round, must be "happy" in at least $\lfloor \alpha T \rfloor$ rounds, meaning that at least one member of S approves the decision (but this member of S can differ across rounds).

Definition 3.1 (PJR)**.** A decision sequence D satisfies *PJR* if for every $\ell \in \mathbb{N}$ and every group of voters $S \subseteq N$ that agrees in every round and has size $|S| \geqslant \ell \cdot \frac{n}{T}$, there are at least ℓ rounds $j \in R$ in which the decision d_j of D is approved by at least one voter in S (i.e., $d_j \in \bigcup_{i\in S} A^i_j$).

[1] We use different names for these axioms. Bulteau et al. (2021) say "all periods intersection PJR" for what we just call PJR, and say "some periods intersection PJR" for what we call Strong PJR.

Round	1 & 2	3 & 4	5 & 6	7 & 8
Voter 1	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
Voter 2	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$
Voter 3	$\{d\}$	$\{d\}$	$\{d\}$	$\{e\}$
Voter 4	$\{d\}$	$\{d\}$	$\{d\}$	$\{f\}$

Figure 2: Example illustrating our axioms.

PJR only provides guarantees for groups that agree in *all* rounds. Strong PJR also gives guarantees for groups that agree only in some of the rounds, though the groups need to be larger: a group that agrees in k rounds deserves to be happy with ℓ decisions if the size of the group is at least $\ell \cdot \frac{n}{k}$ (as compared to $\ell \cdot \frac{n}{T}$ for groups that agree in all rounds). Note that Strong PJR implies PJR (take $k = T$).

Definition 3.2 (Strong PJR)**.** A decision sequence D satisfies *Strong PJR* if for every $\ell \in \mathbb{N}$ and every group of voters $S \subseteq N$ that agrees in k rounds and has size $|S| \geqslant \ell \cdot \frac{n}{k}$, there are at least ℓ rounds $j \in R$ in which the decision d_j of D is approved by at least one voter in S (i.e., $d_j \in \bigcup_{i\in S} A^i_j$).

An equivalent way of stating this axiom is that an α fraction of the voters who agree in a β fraction of the rounds need to be "happy" with at least an $\lfloor \alpha \cdot \beta \rfloor$ fraction of the decisions. For example, consider a group $S \subseteq N$ of voters with fixed size $\ell \cdot \frac{n}{T}$, and let us ask what Strong PJR guarantees for this group. If S agrees in all rounds, then it says that S should be happy with ℓ decisions. If S agrees in $T/2$ rounds, then it says that S should be happy with $\lfloor \frac{\ell}{2} \rfloor$ decisions.

To illustrate the axioms, consider the instance shown in Figure 2, with $T = 8$ rounds and 4 voters. The group $S = \{1, 2\}$ agrees in all rounds (always approving a). Thus, to satisfy PJR, in at least $\ell = 4$ rounds the outcome needs to be either a, b, or c (because $|S| \geqslant \ell \cdot \frac{4}{8}$). The group $S' = \{3, 4\}$ agrees in the first 6 rounds (approving d), so with $\ell' = 3$, because $|S'| \geqslant \ell' \cdot \frac{4}{6}$, Strong PJR requires that in at least 3 rounds, the outcome is either d, e, or f.

A weakness of PJR and Strong PJR is in how they define S being "happy" with a decision ("at least one member of S approves the decision"). This definition can be satisfied by a decision sequence that gives each member of S a utility that is much lower than ℓ (Peters and Skowron 2020, Sec. 4.2). In the example of Figure 2, the decision sequence (b, b, c, c, d, d, e, f) satisfies Strong PJR, but the first two voters each only approve the decision in 2 rounds, instead of in 4 rounds. Following Aziz et al. (2017), we can fix this by defining the axioms EJR (Extended Justified Representation) and Strong EJR which require that at least one member i of S must approve at least ℓ of the decisions, i.e., must have utility $U^i_D \geqslant \ell$. In the example, the decision sequence (d, d, d, d, a, a, a, a) satisfies Strong EJR.

Definition 3.3 (EJR)**.** A decision sequence D satisfies *EJR* if for every $\ell \in \mathbb{N}$ and every group of voters $S \subseteq N$ that agrees in all rounds and that has size $|S| \geqslant \ell \frac{n}{T}$, there is a voter $i \in S$ who approves at least ℓ decisions in D, i.e., $U^i_D \geqslant \ell$.

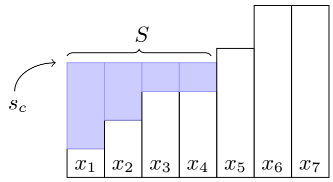

Figure 3: Load distribution

Definition 3.4 (Strong EJR)**.** A decision sequence D satisfies *Strong EJR* if for every $\ell \in \mathbb{N}$ and every group $S \subseteq N$ that agrees in k rounds and has size $|S| \geqslant \ell \cdot \frac{n}{k}$, there is a voter $i \in S$ who approves at least ℓ decisions in D, i.e., $U_D^i \geqslant \ell$.

A decision rule f *satisfies* an axioms if its output satisfies it for all possible inputs. Note that if f is also online, this means that the proportionality guarantee thereby not only holds for the entire decision sequence, but also for every prefix of it.

3.3 Methods

We now define three decision rules that are natural analogues of rules that were first proposed for multi-winner elections.

Sequential Phragmén Phragmén (1894) proposed an approval-based voting method for electing members of the Swedish parliament. Lackner and Maly (2023) adapted this rule to the context of perpetual voting, calling their adaptation "perpetual Phragmén". We follow their definition. The rule makes decisions round by round. Each decision provides a value (or *load*) of 1, which is distributed to voters who approve the decision. In each round, the rule chooses an alternative such that no voter ends up with too much load, thereby prioritizing voters who do not yet agree with many prior decisions. Formally, each voter i starts with load $x^i = 0$. Sequential Phragmén chooses the alternative for which it can distribute a load of 1 in a way that minimizes the maximum total load assigned to a voter: at each round $j \in R$, we compute the following value for each alternative $c \in C_j$:

$$s_c = \min_{S \subseteq \{i \in N : c \in A_j^i\}} \frac{\sum_{i \in S} x^i + 1}{|S|}.$$

This value can be understood using a "water filling" analogy, as shown in Figure 3, where we consider an alternative c approved by 7 voters, and show their current loads as bars. We then fill 1 unit of water on top of the approving voters' loads. Note that the water never falls on top of the loads of voters 5, 6, and 7 because their load is already quite high; in other words, this process has only assigned load to the set $S = \{1, 2, 3, 4\}$. Then s_c is the "water line", which is the load of each voter in S after the load of c has been assigned.[2]

[2]In the multi-winner setting, we can fix S in the definition of s_c to be the set of *all* voters approving c, because it cannot happen that an approver of c already has more load than s_c, for in that case the rule would have chosen c in an earlier iteration (Brill et al. 2023a, Lemma 4.5). This is not true in our setting because c may not have existed in a prior round. Other definitions of Phragmén's method based on virtual bank accounts (Janson 2016; Peters and Skowron 2020) do not easily adapt to our setting due to a similar problem.

The decision for round $j \in R$ is the alternative c that minimizes s_c, breaking ties arbitrarily. After making the decision, we update loads by setting $x^i = s_c$ for each voter $i \in S$ (where S is the coalition attaining the minimum in the definition of s_c) and leave x^i unchanged for voters not in S. Note that the load of a voter will never decrease.

Clearly, Sequential Phragmén is an online rule. Lackner and Maly (2023) show that it can be computed in polynomial time, and provide a detailed example.

Method of Equal Shares (MES) MES is a recently introduced rule for multi-winner voting (Peters and Skowron 2020) and also used in practice for participatory budgeting (Peters, Pierczyński, and Skowron 2021). It can be adapted to our setting in a semi-online fashion: the rule needs to know the total number of rounds T in advance, but does not need to know voter preferences of future rounds. Each decision costs $p = \frac{n}{T}$ units, which must be paid by voters that approve the chosen alternative. MES works by subtracting this amount from an initial budget $b_i = 1$ assigned to each voter i.

For $\rho \geqslant 0$, an alternative $c \in C_j$ is called *ρ-affordable* if

$$\textstyle\sum_{i \in N : c \in A_j^i} \min(b_i, \rho) \geqslant p,$$

so approvers of c can pay p with no one paying more than ρ. In each round, the alternative $d_j \in C_j$ that is ρ-affordable for minimum ρ is chosen, breaking ties arbitrarily. Then for each voter i approving d_j, the remaining budget b_i of i is set to $\max(0, b_i - \rho)$.

If in some round, no alternative is affordable for any ρ, MES *terminates prematurely*. Decisions for the remaining rounds can be made arbitrarily (as far as our axioms are concerned), but in practice are done using an appropriate *completion rule* such as Sequential Phragmén. In our experiments, we followed the "ε-completion" strategy introduced by Peters, Pierczyński, and Skowron (2021, Sec. 3.4) which approximates the concept of choosing the lowest affordable alternative, retaining the core idea of MES.

Proportional Approval Voting (PAV) PAV (based on a rule of Thiele 1895) is an offline rule that selects the decision sequence $D \in C_1 \times \cdots \times C_T$ that maximizes

$$\text{PAV-score}(D) = \sum_{i \in N} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{U_D^i},$$

where we recall that U_D^i is the number of rounds in which voter i approves the decision of D. This harmonic objective function is the unique additive objective that leads to a proportional rule (Aziz et al. 2017). Finding the optimum decision sequence for PAV is NP-hard (Brill et al. 2022, Thm. 5.1), just like in the multi-winner setting (Aziz et al. 2015), but we will show that a polynomial-time local search variant (Aziz et al. 2018) satisfies the same axioms as PAV.[3]

[3]One could use a sequential version of PAV to get an online rule, which Page, Shapiro, and Talmon (2020) conjectured to satisfy at least a weak version of PJR. However, a counterexample from multiwinner voting (Sánchez-Fernández et al. 2017, Table 2) can be adapted to show that Sequential PAV fails PJR in our model (repeat the profile from their paper for $k = 6$ rounds).

4 Satisfying the Axioms

In this section, we establish which of our three rules satisfy which of our four axioms (see Table 1). Bulteau et al. (2021) conjectured that no polynomial-time rule achieves PJR (they gave existence proofs which were based on exponential-time algorithms). Our results show that, in fact, PJR as well as the stronger axioms can be satisfied in polynomial time, by attractive rules, some of which even work online.

4.1 Online: Sequential Phragmén

We begin by analyzing the sequential Phragmén rule.

Theorem 4.1. *Sequential Phragmén satisfies Strong PJR.*

Proof. For a contradiction, suppose $S \subseteq N$ witnesses a violation of Strong PJR: it agrees in rounds $R^* = \{j_1, \dots, j_k\} \subseteq R$ and has size $|S| \geqslant \ell \cdot \frac{n}{k}$, but there are fewer than ℓ rounds in which at least one member of S approves the decision.

First, we claim that if $d_j \in C_j$ is the alternative chosen in some round $j \in R^*$, then $s_{d_j} \leqslant \frac{\ell}{|S|}$. Note that the total load $\sum_{i \in S} x^i$ assigned to members of S is at most $\ell - 1$ because each decision incurs a load of 1. Since $j \in R^*$, there is an alternative $c' \in C_j$ that everyone in S approves. Hence,

$$s_{c'} \leqslant \frac{1 + \sum_{i \in S} x^i}{|S|} \leqslant \frac{1 + (\ell - 1)}{|S|} = \frac{\ell}{|S|},$$

where the first inequality follows from the definition of $s_{c'}$ as a minimum. Since Sequential Phragmén selected d_j, we have $s_{d_j} \leqslant s_{c'}$, showing our claim.

Call a round j a *bad round* if $j \in R^*$ and the decision d_j is not approved by any voter in S. Fix some voter $i \in N \setminus S$, and suppose i gets assigned some load during at least one bad round. Consider the point just after the last bad round j where i is assigned some load. At this point we must have $x^i \leqslant \frac{\ell}{|S|}$ since otherwise $s_{d_j} > \frac{\ell}{|S|}$, contradicting our claim. Thus, at most $\frac{\ell}{|S|}$ load was assigned to i during all bad rounds together. Clearly, this last claim is also true for voters $i \in N \setminus S$ who do not get assigned any load during any bad round.

In a bad round, load is only assigned to voters outside S (since the round is bad). Thus, by summing over all $i \in N \setminus S$, we see that the total load assigned in bad rounds is at most

$$|N \setminus S| \cdot \tfrac{\ell}{|S|} = \tfrac{|N|}{|S|} \cdot \ell - \tfrac{|S|}{|S|} \cdot \ell \leqslant \tfrac{k}{\ell}\ell - \ell = k - \ell.$$

But there are at least $k-(\ell-1)$ bad rounds, so a total load of at least $k - \ell + 1$ is distributed across them, a contradiction. $\square$

Sequential Phragmén fails EJR in multi-winner voting (Brill et al. 2023a), and it also does so in our setting.

Theorem 4.2. *Sequential Phragmén fails EJR.*

4.2 Semi-online: Method of Equal Shares (MES)

We do not know an online rule that satisfies EJR. In fact, in the full version of the paper, we give a reduction showing that such a rule could be converted to a multi-winner voting rule satisfying EJR and the axiom of *committee monotonicity*, the existence of which is a major open problem (Lackner and Skowron 2023; Sánchez-Fernández and Fisteus 2019).

Can we evade this difficulty with some foresight, by relaxing the online requirement? Knowing the time horizon T is a common assumption found in online learning settings like multi-armed bandits (Barman et al. 2023). Indeed, if we know the total number of rounds T, we can use MES (which is online except that it uses T to determine the price $p = n/T$ of deciding a round). We show that it satisfies EJR.

Theorem 4.3. *MES satisfies EJR.*

Does MES provide good guarantees for coalitions that do not agree on all rounds? Unfortunately not. We show that MES fails Strong PJR and Strong EJR. This is perhaps surprising since, in other settings, MES usually satisfies at least as many proportionality axioms as Sequential Phragmén. The reason for its failure here is that coalitions may agree only in early rounds where MES greedily maximizes efficiency, and then MES cannot satisfy the fairness requirements in subsequent rounds where there may not be enough agreement between voters to support the purchase of any alternative.

Theorem 4.4. *MES fails Strong PJR.*

4.3 Offline: Proportional Approval Voting (PAV)

In some settings, offline voting (where alternatives and approval sets for all rounds are known in advance) is possible, e.g., for voting in combinatorial domains with independent issues. Studying the offline setting can also clarify which axioms are plausible aims for online rules. It turns out that if we make decisions fully offline, there is a rule that satisfies all four of our axioms: PAV, as well as a polynomial-time local search variant of PAV. The proof uses a swapping argument: if the output violates Strong EJR, then in at least 1 round, one can change the decision and thereby increase the PAV objective function. This technique is also used in multi-winner voting (Aziz et al. 2017). Our theorem was recently generalized by Masařík, Pierczyński, and Skowron (2023).

Theorem 4.5. *PAV and Local-Search PAV satisfy Strong EJR.*

5 Impossibility of Stronger Guarantees

In this section, we show that if the Strong PJR axiom is further strengthened by reducing the requirement on the coalition size by ε, there are instances where no decision sequence satisfies the strengthening.

Theorem 5.1. *Let $\varepsilon > 0$. Then there exists an instance where no decision sequence D satisfies "ε-*Strong PJR*", defined to require that for every $\ell \in \mathbb{N}$ and every group of voters $S \subseteq N$ that agrees in k rounds and has size $|S| \geqslant (\ell - \varepsilon) \cdot \frac{n}{k}$, there are at least ℓ rounds $j \in R$ in which the decision d_j of D is approved by at least one voter in S (i.e., $d_j \in \bigcup_{i \in S} A_j^i$).*

In the proof, we construct a counterexample where in the first k rounds, there are very many coalitions that agree (but cannot all be satisfied simultaneously), while in the remaining rounds, there is no agreement at all (which makes it impossible to satisfy all the justified demands from earlier rounds). This is a worst case result, and we could hope for rules that satisfy ε-Strong PJR on inputs where it is possible. However, we show that no (semi-)online rule can do this.

Theorem 5.2. *Let $\varepsilon > 0$. No semi-online decision rule returns an ε-*Strong PJR *decision sequence whenever one exists.*

6 Experiments

To understand the performance of our methods empirically, we run our methods on both synthetic and real-world datasets. In addition to our proposed rules, we also consider two rules proposed by Lackner (2020): *Perpetual Quota* (aims at granting each voter a satisfaction as close as possible to their "quota") and *Perpetual Consensus* (similar to Sequential Phragmén but strictly enforces an equal distribution of the load incurred).[4] We chose those rules as they performed well in Lackner's (2020) experiments. Further, we consider two baselines: *Approval Voting* (chooses the alternative with highest approval score in each round) and *Round Robin* (in each round j, voter $j \bmod T$ chooses an alternative).

We evaluate our rules on several metrics of voter utility to complement our theoretical guarantees. For comparability of results, we normalize utility and define a single voter i's utility as the *fraction* of rounds in which the voter approves the decision: U_D^i/T. Based on this, we report three metrics:

- *Average Utility* of the voters (utilitarian social welfare).
- *Utility of the 25th Percentile*: We sort the vector of utilities and report its 25th percentile. This is inspired by egalitarian social welfare, which we did not use in our experiments because the minimum utility was often zero.
- *Gini Coefficient*: This metric quantifies the level of inequality in the voter utilities. A lower value corresponds to a more equal utility distribution (with 0 being obtained in case every voter has the same utility). Formally, the Gini coefficient of a decision sequence D is as follows:

$$\text{gini}(D) = \frac{1}{\sum_{i \in N} U_D^i} \sum_{i \in N} \sum_{j \in N} \frac{1}{2n} \left| U_D^i - D_D^j \right|.$$

6.1 Synthetic Data

For analysis based on synthetic data, we follow a similar setup to the one used by Lackner (2020) which is based on the popular approach of sampling both voters and alternatives as points in a two-dimensional Euclidean space (Elkind et al. 2017). We use $n = 20$ voters who are split into a group of 6 and a group of 14 voters. The locations of voters in the first and second group are sampled from $\mathcal{N}(-0.5, -0.5)$ and $\mathcal{N}(0.5, 0.5)$ respectively, with standard deviation $\sigma = 0.2$. While the voter locations stay fixed across rounds, a fresh set of alternatives is sampled in each round, uniformly at random from the square $[-1, -1] \times [1, 1]$.

We use $T = 20$ rounds with 20 alternatives per round. Each voter approves all alternatives whose Euclidean distance is at most 1.5 times the distance of the closest alternative. On average, voters approved their top 2 alternatives. The experiment was repeated for 1000 trials. We report results for other parameters and voter distributions in the full version.

Across the various distributions, we find some prominent patterns for each metric. In terms of average satisfaction, Approval Voting (AV) outperforms other rules as it maximizes this value. Round Robin performs the worst. Among the proportional rules, we found the following rank order in most cases for average satisfaction: PAV > Sequential Phragmén > MES > Perpetual Quota > Perpetual Consensus. Regarding fairness metric (Gini coefficient), we observed that AV consistently featured a high Gini coefficient and often resulted in almost no decisions being approved by the bottom 25% of the voters. Round Robin performed better than AV and competitive to the proportional rules, which consistently produced outcomes with a more equal distribution of utility.

[4]These rules aim for proportionality, but do not satisfy any of our axioms (Lackner and Maly 2023).

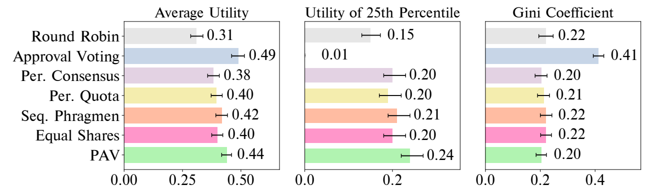

Figure 4: Performance of the rules on synthetic instances. The length of the bar shows the median across the 1000 trials, the error bars give the 25th and 75th percentile, and the numeric text gives the mean.

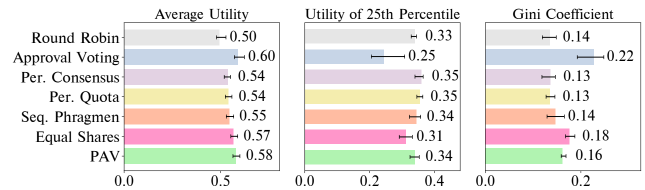

Figure 5: Performance of the rules on the 2022 General Elections in Shasta County, California. The length of the bar shows the median across all voting districts, the error bars give the 25th and 75th percentile, and the text gives the mean.

6.2 Political Data

In addition to synthetic data, we evaluated the rules on data from U.S. political general elections which we collected (available at https://osf.io/t6p7s/). In these elections, voters elect candidates to various federal, state, and local political offices and express opinions on yes/no ballot initiatives. Using public anonymized *Cast Vote Records (CVR)* data we can see, for each voter, their votes on all these issues simultaneously.

We collected instances from 16 counties in California and Colorado from 2020 and 2022 (for which data was available). The results for one county are shown in Figure 5, where Approval Voting (AV), the method that is actually used to determine winners, gives a high Gini coefficient compared to all the other rules (see the full version for more details).

6.3 Learning Preferences from the Moral Machine

Virtual democracy (Noothigattu et al. 2018; Freedman et al. 2020; Mohsin et al. 2021) is a proposal to automate decision making by learning models of preferences of individual users and using predicted preferences as inputs to a voting rule. This can be particularly useful when preference elicitation costs are high, or decisions need to be made in real time. A common approach (Noothigattu et al. 2018; Wang et al. 2019) for virtual democracy is to *average* the learnt model parameters across voters to obtain a single model. Feffer, Heidari, and Lipton (2023) have shown that such averaging may lead to non-proportional outcomes that underweight minority preferences. We hope to alleviate these fairness concerns by aggregating model outputs using our voting rules. Further, we wish to compare our strategy of proportional aggregation with the traditional Machine Learning approach of learning a single preference model on a combined dataset.

Following the work of Noothigattu et al. (2018), to empirically test our work on a dataset that has structured features which allow preference learning, we consider virtual democracy applied to the *moral machine* (Awad et al. 2018). This experiment is a modern take on ethical "trolley problems" and involves decisions that a self-driving car might face. Users were asked to express preferences in instances where a self-driving car must either swerve or stay in the lane, with both choices leading to injuring a different group of people. These choices can be seen as alternatives that can be described by a structured feature vector. Several million pairwise comparison responses are available in a public dataset.

Noothigattu et al. (2018) learn a model predicting the preferences of each respondent. However, there are only 13 pairwise comparisons per respondent, meaning that such individual models have low accuracy. Instead, we partition the respondents by their country. Kim et al. (2018) show that learning a single model for respondents from a country leads to reasonable accuracy (perhaps due to cultural similarities).

We limit ourselves to 197 countries for which the dataset contains over 100 samples. To learn a preference model for each country, we use the Plackett-Luce (PL) model (Plackett 1975; Duncan 1959) which is a *random utility model* appropriate for social choice preference learning (Azari Soufiani, Parkes, and Xia 2012). As a baseline, we train a *combined model* on respondents from all countries, using Additionally, we consider Round Robin which in each round selects a specific country model as the "dictator" whose preferred choice becomes the decision for that round.

We produce 100 decision rounds together with 100 alternatives for each round. We specifically generate alternatives that feature high disagreement (since sampling an alternative uniformly at random typically leads to very similar preferences). We let each country approve the (predicted) best of these alternatives. We then use our voting rules to compute decision outcomes using the country models as voters. To compare the voting rules to the performance of the combined model, we pick the alternative assigned the highest utility by the combined model as its decision for the round.

We present our results in Figure 6 (with results using other parameter settings shown in the full version). We find it striking how Approval Voting (AV) and the Combined Model attain almost identical values on each metric, and how these are quite different from the values obtained by the proportional rules. Indeed, in the experiment, AV and the Combined Model choose the same decision in 84% of rounds, but both agree with the 5 proportional rules less frequently. Notably, the 5 proportional rules all feature a much smaller Gini coefficient and a higher satisfaction at the 25th percentile. The similar performance between AV and the Combined Model suggests that the Combined Model exhibits a bias towards plurality and majority opinions. This contradicts a possible

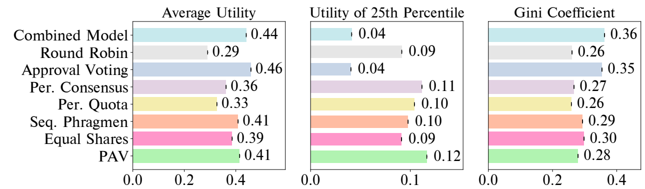

Figure 6: Performance of the different rules on the Moral Machine Dataset, based on high-disagreement alternatives, for $T = 100$ rounds with each round having 100 alternatives and each voter approving their top 1 alternative.

hope one might have had that, because the Combined Model is trained based on an equal number of samples from each country, it will "merge" their views roughly proportionally.

We see the results from this small experiment as a potential starting point for a larger research program that studies how a global preference model trained on preferences of diverse agents makes decisions compared to aggregation rules that explicitly take into account each individual's preferences. Note that the results do not indicate such models can be used in applications, or to automate moral reasoning, which would require more rigorous testing and ethical considerations.

7 Discussion and Future Work

Extensions Our model can be extended to make it compatible with more real-world applications. Examples include allowing weighting issues by importance (Page, Shapiro, and Talmon 2020), to allow for dependencies between issues (Brill et al. 2023b), or to allow voters to specify utilities or to rank alternatives instead of approvals (Peters, Pierczyński, and Skowron 2021). The latter may be important as some issues may be more critical to certain minority groups.

Open Problems We leave some theoretical open problems for future work, notably whether an online rule can satisfy EJR or Strong EJR – a negative result may be easier to find in this setting than for multi-winner voting. Also open is whether a semi-online rule can satisfy Strong EJR. More conceptually, are there stronger versions of EJR for this setting that are still satisfiable? Can the concept of proportionality degree (Skowron 2021) be adapted to our setting? What about FJR (Full Justified Representation) or the core?

Strategic Issues Peters (2018) proved an impossibility theorem showing that no proportional multi-winner voting rule can be *strategyproof*: voters may be able to get a better outcome by misrepresenting their preferences. For the special case of "approval-based apportionment" (Brill et al. 2022; Airiau et al. 2023), the impossibility still holds (Delemazure et al. 2023; Lackner, Maly, and Nardi 2023), Because this is also a special case of our model (when the set of alternatives and voter preferences are the same in each round), it follows that no proportional rule in our setting can be strategyproof. Further, the order in which the set of issues are presented can also change the outcomes of online rules and this may present another possibility for manipulation.

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References

Airiau, S.; Aziz, H.; Caragiannis, I.; Kruger, J.; Lang, J.; and Peters, D. 2023. Portioning using ordinal preferences: Fairness and efficiency. *Artificial Intelligence*, 314: 103809.

Amanatidis, G.; Barrot, N.; Lang, J.; Markakis, E.; and Ries, B. 2015. Multiple referenda and multiwinner elections using Hamming distances: Complexity and manipulability. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 715–723.

Awad, E.; Dsouza, S.; Kim, R.; Schulz, J.; Henrich, J.; Shariff, A.; Bonnefon, J.-F.; and Rahwan, I. 2018. The moral machine experiment. *Nature*, 563(7729): 59–64.

Azari Soufiani, H.; Parkes, D. C.; and Xia, L. 2012. Random utility theory for social choice. *Advances in Neural Information Processing Systems*, 25: 126–134.

Aziz, H.; Brill, M.; Conitzer, V.; Elkind, E.; Freeman, R.; and Walsh, T. 2017. Justified Representation in approval-based committee voting. *Social Choice and Welfare*, 48(2): 461–485.

Aziz, H.; Elkind, E.; Huang, S.; Lackner, M.; Sánchez-Fernández, L.; and Skowron, P. 2018. On the complexity of extended and proportional justified representation. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*, 902–909.

Aziz, H.; Gaspers, S.; Gudmundsson, J.; Mackenzie, S.; Mattei, N.; and Walsh, T. 2015. Computational aspects of multi-winner approval voting. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 107–115.

Barman, S.; Khan, A.; Maiti, A.; and Sawarni, A. 2023. Fairness and welfare quantification for regret in multi-armed bandits. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*, 6762–6769.

Barrot, N.; and Lang, J. 2016. Conditional and sequential approval voting on combinatorial domains. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, 88–94.

Barrot, N.; Lang, J.; and Yokoo, M. 2017. Manipulation of Hamming-based approval voting for multiple referenda and committee elections. In *Proceedings of the 16th Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 597–605.

Brill, M.; Freeman, R.; Janson, S.; and Lackner, M. 2023a. Phragmén's voting methods and justified representation. *Mathematical Programming*.

Brill, M.; Gölz, P.; Peters, D.; Schmidt-Kraepelin, U.; and Wilker, K. 2022. Approval-based apportionment. *Mathematical Programming*.

Brill, M.; Markakis, E.; Papasotiropoulos, G.; and Peters, J. 2023b. Proportionality guarantees in elections with interdependent issues. In *Proceedings of the 32nd International Joint Conference on Artificial Intelligence (IJCAI)*, 2537–2545.

Bulteau, L.; Hazon, N.; Page, R.; Rosenfeld, A.; and Talmon, N. 2021. Justified representation for perpetual voting. *IEEE Access*, 9: 96598–96612.

Chandak, N.; Goel, S.; and Peters, D. 2023. Proportional aggregation of preferences for sequential decision making. *arXiv:2306.14858*.

Conitzer, V.; Freeman, R.; and Shah, N. 2017. Fair public decision making. In *Proceedings of the 2017 ACM Conference on Economics and Computation (EC)*, 629–646.

Delemazure, T.; Demeulemeester, T.; Eberl, M.; Israel, J.; and Lederer, P. 2023. Strategyproofness and proportionality in party-approval multiwinner elections. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*, 5591–5599.

Duncan, L. R. 1959. *Individual choice behavior: A theoretical analysis*. New York: Wiley.

Elkind, E.; Faliszewski, P.; Laslier, J.-F.; Skowron, P.; Slinko, A.; and Talmon, N. 2017. What do multiwinner voting rules do? An experiment over the two-dimensional Euclidean domain. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, 494–501.

Feffer, M.; Heidari, H.; and Lipton, Z. 2023. Moral machine or tyranny of the majority? In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*, 5974–5982.

Freedman, R.; Borg, J. S.; Sinnott-Armstrong, W.; Dickerson, J. P.; and Conitzer, V. 2020. Adapting a kidney exchange algorithm to align with human values. *Artificial Intelligence*, 283: 103261.

Freeman, R.; Kahng, A.; and Pennock, D. M. 2020. Proportionality in approval-based elections with a variable number of winners. In *Proceedings of the 29th International Joint Conferences on Artificial Intelligence (IJCAI)*, 132–138.

Janson, S. 2016. Phragmén's and Thiele's election methods. *arXiv:1611.08826*.

Kim, R.; Kleiman-Weiner, M.; Abeliuk, A.; Awad, E.; Dsouza, S.; Tenenbaum, J. B.; and Rahwan, I. 2018. A computational model of commonsense moral decision making. In *Proceedings of the 2018 AAAI/ACM Conference on AI, Ethics, and Society*, 197–203.

Lackner, M. 2020. Perpetual voting: Fairness in long-term decision making. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*, 2103–2110.

Lackner, M.; and Maly, J. 2023. Proportional decisions in perpetual voting. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*, 5722–5729.

Lackner, M.; Maly, J.; and Nardi, O. 2023. Free-riding in multi-issue decisions. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2040–2048.

Lackner, M.; and Skowron, P. 2023. *Multi-Winner Voting with Approval Preferences*. Springer.

Lang, J.; and Xia, L. 2016. Voting in Combinatorial Domains. In Brandt, F.; Conitzer, V.; Endriss, U.; Lang, J.; and Procaccia, A. D., eds., *Handbook of Computational Social Choice*, chapter 9. Cambridge University Press.

Lee, M. K.; Kusbit, D.; Kahng, A.; Kim, J. T.; Yuan, X.; Chan, A.; See, D.; Noothigattu, R.; Lee, S.; Psomas, A.; and Procaccia, A. D. 2019. WeBuildAI: Participatory framework for algorithmic governance. *Proceedings of the ACM on Human-Computer Interaction (HCI)*, 3.

Masařík, T.; Pierczyński, G.; and Skowron, P. 2023. A generalised theory of proportionality in collective decision making. *arXiv:2307.06077*.

Mohsin, F.; Luo, L.; Ma, W.; Kang, I.; Zhao, Z.; Liu, A.; Vaish, R.; and Xia, L. 2021. Making group decisions from natural language-based preferences. In *Proceedings of the 8th International Workshop on Computational Social Choice (COMSOC)*.

Noothigattu, R.; Gaikwad, S. S.; Awad, E.; Dsouza, S.; Rahwan, I.; Ravikumar, P.; and Procaccia, A. D. 2018. A voting-based system for ethical decision making. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*, 1587–1594.

Page, R.; Shapiro, E.; and Talmon, N. 2020. Electing the Executive Branch. *arXiv:2009.09734*.

Peters, D. 2018. Proportionality and strategyproofness in multiwinner elections. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 1549–1557.

Peters, D.; Pierczyński, G.; and Skowron, P. 2021. Proportional participatory budgeting with additive utilities. In *Advances in Neural Information Processing Systems*, volume 34, 12726–12737.

Peters, D.; and Skowron, P. 2020. Proportionality and the limits of welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*, 793–794. Full version arXiv:1911.11747.

Phragmén, E. 1894. Sur une méthode nouvelle pour réaliser, dans les élections, la représentation proportionnelle des partis. *Öfversigt af Kongliga Vetenskaps-Akademiens Förhandlingar*, 51(3): 133–137.

Plackett, R. L. 1975. The analysis of permutations. *Journal of the Royal Statistical Society Series C: Applied Statistics*, 24(2): 193–202.

Sánchez-Fernández, L.; Elkind, E.; Lackner, M.; Fernández, N.; Fisteus, J.; Val, P. B.; and Skowron, P. 2017. Proportional justified representation. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, 670–676.

Sánchez-Fernández, L.; and Fisteus, J. A. 2019. Monotonicity axioms in approval-based multi-winner voting rules. In *Proceedings of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 485–493.

Skowron, P. 2021. Proportionality degree of multiwinner rules. In *Proceedings of the 22nd ACM Conference on Economics and Computation (EC)*, 820–840.

Skowron, P.; and Górecki, A. 2022. Proportional public decisions. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*, 5191–5198.

Thiele, T. N. 1895. Om Flerfold Valg. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Fordhandlinger*.

Wang, T.; Zhao, J.; Yu, H.; Liu, J.; Yang, X.; Ren, X.; and Shi, S. 2019. Privacy-preserving crowd-guided AI decision-making in ethical dilemmas. In *Proceedings of the 28th ACM International Conference on Information and Knowledge Management*, 1311–1320.