Simple Causes of Complexity in Hedonic Games

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one reduction to rule them all*

"Finding friends is hard, and there is not much we can do about that."
"The last NP-hardness reductions concerning stability in hedonic games."

Hedonic Games: The Model

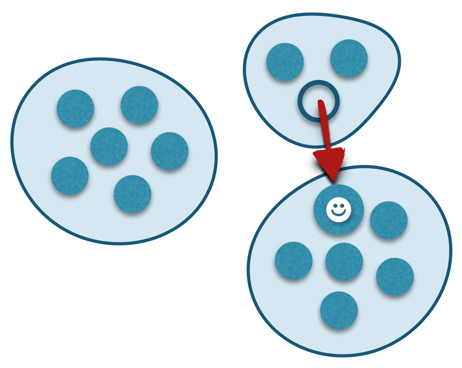
Finite set N of agents, each $i \in N$ having preferences $\succcurlyeq_i$ over groups of agents:

$$\{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\} \succ_1 \{1,3\}$$

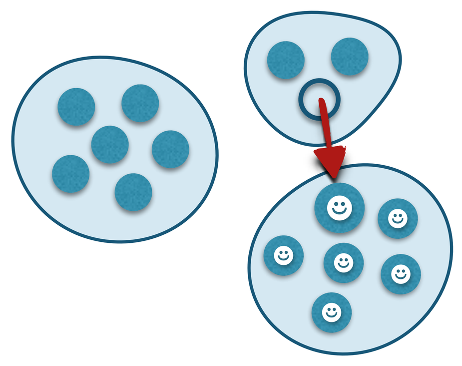
Outcome: a partition π of the agent set N.

Stability Concepts

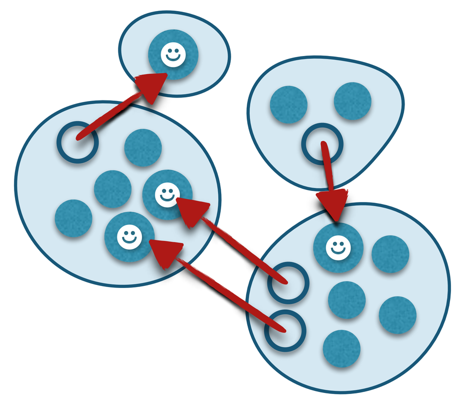
We want partition π to be *stable*. What could this mean?



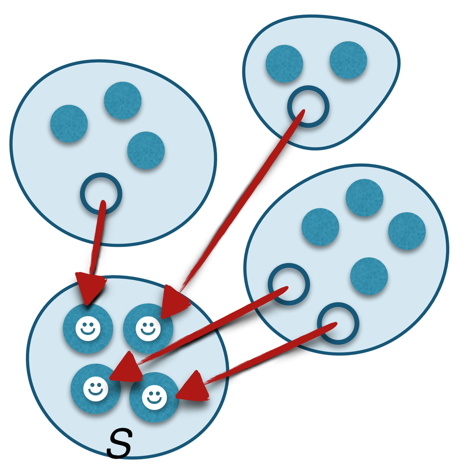
Nash stable (NS): no individual wants to change into another group.



Individually stable (IS): no individual can change into a group in which all members welcome her.



Strongly Nash stable (SNS): no group of agents want to simultaneously change into other group(s). Our hardness results are the first for this concept.



Core stable (CR): no group S of agents all prefer S to where they are in π.

Strict Core stable (SCR): only require 1 deviator in S to have a strict preference.

Classes of Hedonic Games

Every agent needs to order $2^{|N|-1}$ groups. Thus we need to restrict the possible preferences so they admit a concise representation. Standard examples from the literature:

IRCL: list all coalitions preferred to $\{i\}$

HC-nets: describe preferences using Boolean rules

$\mathcal{W}$-games: order agents, and order coalitions by worst agent

ASHG: order coalitions by $\sum_{j \in S} v_i(j)$

FHG: order coalitions by $\frac{1}{|S|} \sum_{j \in S} v_i(j)$

In addition, we introduce several natural new classes:

Median: order coalitions by the median of the $v_i(j)$

Geometric Mean: order coalitions by $\sqrt[|S|]{\prod v_i(j)}$

Nash Product: order coalitions by $\prod_{j \in S} v_i(j)$

Midrange: take the average of best and worst $v_i(j)$

***k*-Approval**: sum the k best $v_i(j)$ in S.

Overview of our Hardness Results

	SNS	SCR	CR	NS	IS
IRCL of length $\leqslant 9$	NP-h.	NP-c.	NP-c.	NP-c.	NP-c.
Hedonic Coalition Nets	NP-h.	NP-h.	NP-h.	NP-c.	NP-c.
$\mathcal{W}$-preferences (no ties)		(P)	(P)	NP-c.	NP-c.
$\mathcal{W}$-preferences	NP-h.		NP-c.	NP-c.	NP-c.
$\mathcal{WB}$-preferences (no ties)		(P)	(P)	NP-c.	NP-c.
$\mathcal{WB}$-preferences	NP-h.		NP-c.	NP-c.	NP-c.
B- & W-hedonic games	NP-h.		NP-h.	NP-c.	NP-c.
Additively separable	NP-h.	NP-h.	NP-h.	NP-c.	NP-c.
Fractional hedonic games	NP-h.	NP-h.	NP-h.	NP-c.	NP-c.
Social FHGs		NP-h.	NP-h.	(+)	(+)
Median		NP-h.	NP-h.		
Midrange ($\frac{1}{2}\mathcal{B} + \frac{1}{2}\mathcal{W}$)	NP-h.		NP-h.	NP-c.	NP-c.
4-Approval	NP-h.	NP-h.	NP-h.	NP-c.	NP-c.

gray: new result. red: settling an open problem. white: known, replicated here.
(P): *known poly-time.* (+): *always exists.*

Let α be a stability concept and $\mathcal{C}$ a class of hedonic games.

α-EXISTENCE FOR $\mathcal{C}$

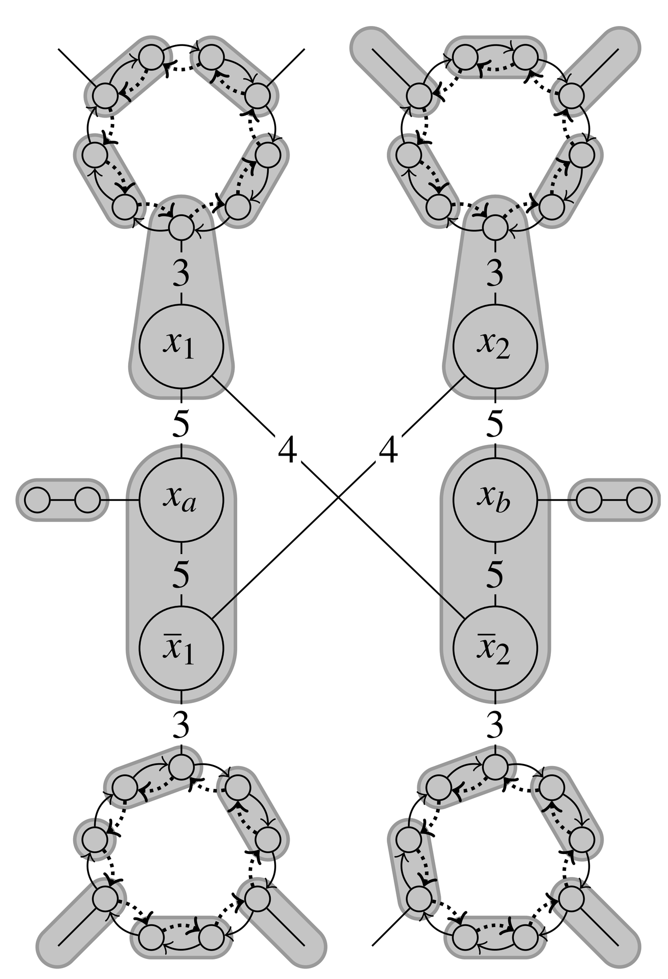
Instance: Hedonic game $\langle N, (\succcurlyeq_i)_{i \in N} \rangle$ from $\mathcal{C}$

Question: Does there exist an α-stable partition π of N?

Theorem. *(Informal Statement)*. The above problem is NP-hard whenever $\mathcal{C}$ includes enough games so that
- agents may order coalitions $\{i,j\}$ arbitrarily, and
- agents may have enemies such that they dislike any coalition containing too many enemies.

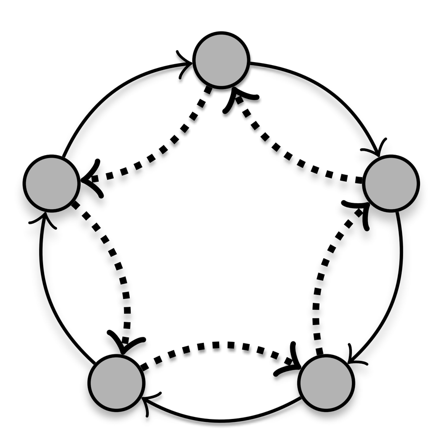
Simple & natural conditions: all classes listed on the left are easily seen to satisfy them. Stronger results need stronger conditions (see paper): 'axiomatic' approach unifies the field.

Problem remains hard even if preferences over $\{i,j\}$ are strict (at least for NS and IS), and even if no agent has more than 3 friends, improving on past X3C reductions using 11 friends.



*actually 3 reductions (from a restricted version of 3SAT).

Key observation: conditions of theorem imply that "odd cycles" do not admit any stable outcome. In the pentagon on the right, each agent prefers her clockwise successor to her predecessor, and is enemies with the two remaining players. We use this game as a clause gadget.



Presented at IJCAI 2015.