COMSOC Lecture 2:
Allocation of Indivisible Items

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Allocation of indivisible items

- $N = \{1, \ldots, n\}$ is a set of agents.
- $O = \{o_1, \ldots, o_m\}$ is a set of items/objects/goods.
- An allocation is a list $A = (A_1, \ldots, A_n)$, where $A_i \subseteq O$ is a bundle of items assigned to agent i. Bundles must be pairwise disjoint. We also must have $A_1 \cup \cdots \cup A_n = O$; if this condition is not satisfied, we speak of a partial allocation.
- Each agent i has a valuation function $v_i : 2^O \to \mathbb{R}_{\geq 0}$ that is monotonic: $B_1 \subseteq B_2 \implies v_i(B_1) \leq v_i(B_2)$. (items are goods)
- A valuation function is additive if $v_i(B) = \sum_{o \in B} v_i(\{o\})$ for all $B \subseteq O$.
  - In this case, we also write $v_i(o) := v_i(\{o\})$.
  - What are some examples of non-additive valuation functions?

Example

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🧔	8	7	**20**	5
👱‍♀️	9	11	12	**8**
🧑‍🦰	**9**	**10**	18	3

Proportionality and envy-freeness

Let A be an allocation.

- A is proportional if $v_i(A_i) \geq \frac{1}{n} v_i(O)$ for every $i \in N$.
- A is envy-free if $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$

Question: are there examples where no envy-free allocation exists? no proportional allocation?

Proportionality and envy-freeness

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Question: are there examples where no envy-free allocation exists? no proportional allocation?

Yes. $N = \{1, 2\}$, $O = \{o_1\}$, $v_1(o_1) = v_2(o_1) = 1$.

- For the allocation $(\{o_1\}, \emptyset)$, 2 envies 1 and doesn't get proportional share.
- For the allocation $(\emptyset, \{o_1\})$, 1 envies 2 and doesn't get proportional share.

Deciding existence

Consider the following decision problem [and variant]:
EXISTENCE OF PROPORTIONAL [ENVY-FREE] ALLOCATION

- **Input**: Additive valuations $(v_i(o))_{i \in N, o \in O}$.
- **Question:** Does there exist a (complete) allocation A that is proportional? [that is envy-free?]

This problem is NP-complete.

Obvious reduction from PARTITION, works even for $n = 2$ agents.

- **Input**: List of numbers $(x_1, \dots, x_m)$
- **Question:** Does there exist a partition (S_1, S_2) of $\{1, \dots, m\}$ such that $\sum_{i \in S_1} x_i = \sum_{i \in S_2} x_i$?

Exercise: This only shows *weak* NP-hardness (binary encoding of numbers). Show the problem is strongly NP-hard (unrestricted n).

Some allocation rules

- Maximize utilitarian social welfare: Pick an allocation A that maximizes $\sum_{i \in N} v_i(A_i)$.
- Maximize egalitarian social welfare: Pick an allocation A that maximizes $\min_{i \in N} v_i(A_i)$.
- Maximize Nash social welfare: Pick an allocation A that maximizes $\prod_{i \in N} v_i(A_i)$.
  - This is the same as maximizing $\sum_{i \in N} \log v_i(A_i)$.
  - This is scale-free: multiplying the valuations of an agent by any factor does not change the optimal allocation.
  - It lies "between" utilitarian and egalitarian social welfare: $\min_{i \in N} v_i(A_i) \leq \sqrt[n]{\prod_{i \in N} v_i(A_i)} \leq \frac{1}{n} \sum_{i \in N} v_i(A_i)$. (AM-GM inequality)

Question: What is the compuational complexity of computing optimal allocations for these objectives?

Example

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Envy-freeness up to 1 good (EF1)

An allocation is envy-free up to 1 good (EF1) if for all $i, j \in N$,

either $v_i(A_i) \geq v_i(A_j)$ or there is $o \in A_j$ with $v_i(A_i) \geq v_i(A_j \setminus \{o\})$.

Exercise: Find an EF1 allocation:

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👱‍♀️	9	11	12	8
👨‍🦰	9	10	18	3

Theorem: An EF1 allocation always exists.

Round robin rule

Consider the following procedure:

> *Repeatedly go through the agents in order (1 2 3 …n 1 2 3 …n 1 2 3 4) and on each agent's turn, let them pick an unpicked good that is most valuable to them.*

- Clearly, this is EF1 for agent 1 (in fact, he is envy-free).
- But it is also EF1 for everyone else. Consider for example agent 3. Let him ignore the first item that agent 1 picked, and the first item that agent 2 picked. With these ignored, no envy remains.

Question: what are some other agent orderings that guarantee EF1? what are some that don't?

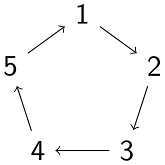
Question: Does this algorithm work for non-additive valuations?

Envy graph, cycle elimination

Given an allocation A, its envy graph is the directed graph with 1 vertex for each agent, and an arc from i to j if i envies j.

Consider some allocation A. Suppose the envy graph has a cycle 1-2-3-4-5-1, meaning that

$$
\begin{aligned}
v_1(A_1) &< v_1(A_2) \\
v_2(A_2) &< v_2(A_3) \\
v_3(A_3) &< v_3(A_4) \\
v_4(A_4) &< v_4(A_5) \\
v_5(A_5) &< v_5(A_1).
\end{aligned}
$$



Then we can eliminate the cycle by giving A_2 to A_1, A_3 to A_2, etc. The resulting allocation is does not introduce any additional envy edges (and it is a Pareto improvement). If A was EF1, then same is true after.

Envy graph algorithm

1. Start with the empty (partial) allocation A.
2. For each item $o \leftarrow [o_1, o_2, \ldots, o_m]$, in order:
   - Compute the envy graph for A, and update A by eliminating any cycles.
   - Now the envy graph has no cycles. Pick an agent i who is a source in the envy graph, i.e. is not envied by anybody.
   - Add o to A_i.

Theorem: This algorithm always terminates with an EF1 allocation.

Proof: partial allocation is EF1 throughout. Let A be allocation before adding o, B after. Then

$$v_j(B_j) = v_j(A_j) \overset{i \text{ source}}{\geq} v_j(A_i) = v_j(B_i \setminus \{o\}).$$

Question: Does this algorithm work for non-additive valuations?

Pareto-optimality

An allocation A is Pareto-optimal if there is no other allocation B such that $v_i(A_i) \geq v_i(B_i)$ for all $i \in N$ and $v_i(A_i) > v_i(B_i)$ for some $i \in N$.

Questions: Which rules are Pareto-optimal? Is round robin? Is envy graph?

Pareto-optimality

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Questions: Which rules are Pareto-optimal? Is round robin? Is envy graph?

Question: Does there always exist a Pareto-optimal EF1 allocation?

Maximizing Nash Welfare is PO and EF1

The MNW (Max Nash Welfare) rule selects an allocation maximizing $\prod_{i \in N} v_i(A_i)$.

Clearly, this rule is PO.*

Proved in 2016: it also satisfies EF1.

- Fix any agents $i, j \in N$, and consider moving object $o \in A_j$ from A_j to A_i.
- $v_i(A_i \cup \{o\}) \cdot v_j(A_j \setminus \{o\}) \leq v_i(A_i) \cdot v_j(A_j)$.
- $\Rightarrow$: $1 - v_j(o)/v_j(A_j) \leq 1 - v_i(o)/(v_i(A_i) + v_i(o))$.
- $\Rightarrow$: $v_j(o)/v_j(A_j) \geq v_i(o)/(v_i(A_i) + v_i(o^*))$ for $o^* \in \arg\max_{o' \in A_j} v_i(o')$.
- Sum over all $o \in A_j$.

Maximizing Nash Welfare

- Used on Spliddit
- Can calculate with ILP.
- `https://pref.tools/nash-indivisible/`
- There is a pseudo-polynomial algorithm achieving PO + EF1 (i.e., polynomial in n, m, $\max_{i,o} v_i(o)$).

Is EF1 enough?

	car	balloon	socks
A	100	1	1
B	100	1	1

	car	balloon	socks
A	100	1	1
B	100	1	1

Envy-freeness up to any good (EFX)

Definition: An allocation A satisfies EFX if for all $i, j \in N$, and for any good $o \in A_j$, we have

$$v_i(A_i) \geq v_i(A_j \setminus \{o\})$$

- Open: Does there always exist an EFX allocation?
- Known: exists for two agents (easy), exists for three agents (very hard)
- Known: exists for identical valuations.
  - Method that works for two agents and for identical valuations: leximin
  - Maximize the utility of the worst-off agent. Subject to this, maximize the utility of the second-worst-off agent, etc.

Non-additive valuations?

A valuation function $v_i : 2^O \to \mathbb{R}$ is submodular if for all $A \subseteq B$ and all $x \in O \setminus B$,

$$v_i(B \cup \{o\}) - v_i(B) \leq v_i(A \cup \{o\}) - v_i(A).$$

Example: course allocation.

An EF1 allocation always exists for submodular violation.

- Open: does a PO + EF1 allocation always exist?
  Nash is not EF1.

What about chores?

A chore for agent i is an item with $v_i(o) < 0$.

We can define EF1 for mixed instances as follows:

An allocation A is EF1 if for all $i, j \in N$, there is some object $o \in A_i \cup A_j$ such that

$$v_i(A_i \setminus \{o\}) \geq v_i(A_j \setminus \{o\})$$

- For 2 agents, can do PO + EF1.
- Can always do EF1 (without PO).
- Open: can we do PO + EF1 for 3+ agents? (open even for instances with only chores)